Hyperbolic Function Integration Problem 1

$$\int x^m \operatorname{Tanh}[a+b x]\, dx$$

- *Rubi* returns m+2 term sums for positive integer m:

```
Int[x Tanh[a + b x], x]
```

$$-\frac{x^2}{2}+\frac{x \operatorname{Log}\left[1+e^{2 a+2 b x}\right]}{b}+\frac{\operatorname{PolyLog}\left[2,-e^{2 a+2 b x}\right]}{2 b^2}$$

```
Int[x^2 Tanh[a + b x], x]
```

$$-\frac{x^3}{3}+\frac{x^2 \operatorname{Log}\left[1+e^{2 a+2 b x}\right]}{b}+\frac{x \operatorname{PolyLog}\left[2,-e^{2 a+2 b x}\right]}{b^2}-\frac{\operatorname{PolyLog}\left[3,-e^{2 a+2 b x}\right]}{2 b^3}$$

```
Int[x^3 Tanh[a + b x], x]
```

$$-\frac{x^4}{4}+\frac{x^3 \operatorname{Log}\left[1+e^{2 a+2 b x}\right]}{b}+\frac{3 x^2 \operatorname{PolyLog}\left[2,-e^{2 a+2 b x}\right]}{2 b^2}-\frac{3 x \operatorname{PolyLog}\left[3,-e^{2 a+2 b x}\right]}{2 b^3}+\frac{3 \operatorname{PolyLog}\left[4,-e^{2 a+2 b x}\right]}{4 b^4}$$

- *Mathematica* returns a 10 term sum involving the imaginary unit when m is 1:

$$\int x \operatorname{Tanh}[a+b x]\, dx$$

$$\frac{i \pi x}{2 b}+\frac{x \operatorname{ArcTanh}[\operatorname{Coth}[a]]}{b}-\frac{i \pi \operatorname{Log}\left[1+e^{2 b x}\right]}{2 b^2}+$$
$$\frac{x \operatorname{Log}\left[1-e^{-2 (b x+\operatorname{ArcTanh}[\operatorname{Coth}[a]])}\right]}{b}+\frac{\operatorname{ArcTanh}[\operatorname{Coth}[a]] \operatorname{Log}\left[1-e^{-2 (b x+\operatorname{ArcTanh}[\operatorname{Coth}[a]])}\right]}{b^2}+$$
$$\frac{i \pi \operatorname{Log}[\operatorname{Cosh}[b x]]}{2 b^2}-\frac{\operatorname{ArcTanh}[\operatorname{Coth}[a]] \operatorname{Log}[i \operatorname{Sinh}[b x+\operatorname{ArcTanh}[\operatorname{Coth}[a]]]]}{b^2}-$$
$$\frac{\operatorname{PolyLog}\left[2, e^{-2 (b x+\operatorname{ArcTanh}[\operatorname{Coth}[a]])}\right]}{2 b^2}+\frac{1}{2} x^2 \operatorname{Tanh}[a]-\frac{1}{2} e^{-\operatorname{ArcTanh}[\operatorname{Coth}[a]]} x^2 \sqrt{-\operatorname{Csch}[a]^2} \operatorname{Tanh}[a]$$

$$\int x^2 \operatorname{Tanh}[a+b x]\, dx$$

$$-\frac{x^3}{3}+\frac{x^2 \operatorname{Log}\left[1+e^{2 (a+b x)}\right]}{b}+\frac{x \operatorname{PolyLog}\left[2,-e^{2 (a+b x)}\right]}{b^2}-\frac{\operatorname{PolyLog}\left[3,-e^{2 (a+b x)}\right]}{2 b^3}$$

$$\int x^3 \operatorname{Tanh}[a+b x]\, dx$$

$$-\frac{x^4}{4}+\frac{x^3 \operatorname{Log}\left[1+e^{2 (a+b x)}\right]}{b}+\frac{3 x^2 \operatorname{PolyLog}\left[2,-e^{2 (a+b x)}\right]}{2 b^2}-\frac{3 x \operatorname{PolyLog}\left[3,-e^{2 (a+b x)}\right]}{2 b^3}+\frac{3 \operatorname{PolyLog}\left[4,-e^{2 (a+b x)}\right]}{4 b^4}$$

- *Maple* returns m+5 term sums, 3 of which are superfluous since their derivative is zero:

```
int(x*tanh(a+b*x), x);
```

$$-\frac{a^2}{b^2}-\frac{2 a x}{b}-\frac{x^2}{2}+\frac{2 a \operatorname{Log}\left[e^{a+b x}\right]}{b^2}+\frac{x \operatorname{Log}\left[1+e^{2 a+2 b x}\right]}{b}+\frac{\operatorname{PolyLog}\left[2,-e^{2 a+2 b x}\right]}{2 b^2}$$

```
int(x^2*tanh(a+b*x), x);
```

$$\frac{4\,a^3}{3\,b^3}+\frac{2\,a^2\,x}{b^2}-\frac{x^3}{3}-\frac{2\,a^2\,\mathrm{Log}\left[e^{a+b\,x}\right]}{b^3}+\frac{x^2\,\mathrm{Log}\left[1+e^{2\,a+2\,b\,x}\right]}{b}+\frac{x\,\mathrm{PolyLog}\left[2,-e^{2\,a+2\,b\,x}\right]}{b^2}-\frac{\mathrm{PolyLog}\left[3,-e^{2\,a+2\,b\,x}\right]}{2\,b^3}$$

```
int (x^3 * tanh (a + b * x), x);
```

$$\frac{3\,a^4}{2\,b^4}-\frac{2\,a^3\,x}{b^3}-\frac{x^4}{4}+\frac{2\,a^3\,\mathrm{Log}\left[e^{a+b\,x}\right]}{b^4}+\frac{x^3\,\mathrm{Log}\left[1+e^{2\,a+2\,b\,x}\right]}{b}+$$
$$\frac{3\,x^2\,\mathrm{PolyLog}\left[2,-e^{2\,a+2\,b\,x}\right]}{2\,b^2}-\frac{3\,x\,\mathrm{PolyLog}\left[3,-e^{2\,a+2\,b\,x}\right]}{2\,b^3}+\frac{3\,\mathrm{PolyLog}\left[4,-e^{2\,a+2\,b\,x}\right]}{4\,b^4}$$

Note that these systems give similar results to the above for the hyperbolic cotangent function.

Hyperbolic Function Integration Problem 2

$$\int \frac{x^m}{a+b\,\mathrm{Sinh}[x]}\,dx$$

- *Rubi* returns 2m+2 term sums for positive integer m:

$$\mathrm{Int}\left[\frac{x}{a+b\,\mathrm{Sinh}[x]},x\right]$$

$$\frac{x\,\mathrm{Log}\left[1+\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{x\,\mathrm{Log}\left[1+\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

$$\mathrm{Int}\left[\frac{x^2}{a+b\,\mathrm{Sinh}[x]},x\right]$$

$$\frac{x^2\,\mathrm{Log}\left[1+\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{x^2\,\mathrm{Log}\left[1+\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{2\,x\,\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-$$
$$\frac{2\,x\,\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{2\,\mathrm{PolyLog}\left[3,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{2\,\mathrm{PolyLog}\left[3,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

$$\mathrm{Int}\left[\frac{x^3}{a+b\,\mathrm{Sinh}[x]},x\right]$$

$$\frac{x^3\,\mathrm{Log}\left[1+\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{x^3\,\mathrm{Log}\left[1+\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{3\,x^2\,\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{3\,x^2\,\mathrm{PolyLog}\left[2,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-$$
$$\frac{6\,x\,\mathrm{PolyLog}\left[3,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{6\,x\,\mathrm{PolyLog}\left[3,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{6\,\mathrm{PolyLog}\left[4,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{6\,\mathrm{PolyLog}\left[4,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

- *Mathematica* returns a huge result involving the imaginary unit when m is 1:

$$\int \frac{x}{a+b\,\mathrm{Sinh}[x]}\,dx$$

$$-\frac{i\,\pi\,\text{ArcTanh}\left[\frac{-b+a\,\text{Tanh}\left[\frac{x}{2}\right]}{\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{1}{\sqrt{-a^2-b^2}}$$

$$\left(2\,\text{ArcCos}\left[-\frac{i\,a}{b}\right]\text{ArcTanh}\left[\frac{(a+i\,b)\,\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]}{\sqrt{-a^2-b^2}}\right]+(\pi-2\,i\,x)\,\text{ArcTanh}\left[\frac{(a-i\,b)\,\text{Tan}\left[\frac{1}{4}(\pi+2\,i\,x)\right]}{\sqrt{-a^2-b^2}}\right]-\right.$$

$$\left(\text{ArcCos}\left[-\frac{i\,a}{b}\right]+2\,i\,\text{ArcTanh}\left[\frac{(a+i\,b)\,\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]}{\sqrt{-a^2-b^2}}\right]\right)$$

$$\text{Log}\left[\left((i\,a+b)\left(a+i\left(b+\sqrt{-a^2-b^2}\right)\right)\left(-i+\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]\right)\right)\Big/\right.$$

$$\left.\left(b\left(i\,a+b+i\sqrt{-a^2-b^2}\;\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]\right)\right)\right]-$$

$$\left(\text{ArcCos}\left[-\frac{i\,a}{b}\right]-2\,i\,\text{ArcTanh}\left[\frac{(a+i\,b)\,\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]}{\sqrt{-a^2-b^2}}\right]\right)\text{Log}\Big[$$

$$\left.\left((i\,a+b)\left(i\,a-b+\sqrt{-a^2-b^2}\right)\left(i+\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]\right)\right)\Big/\left(b\left(a-i\,b+\sqrt{-a^2-b^2}\;\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]\right)\right)\right]+$$

$$\left(\text{ArcCos}\left[-\frac{i\,a}{b}\right]-2\,i\,\text{ArcTanh}\left[\frac{(a+i\,b)\,\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]}{\sqrt{-a^2-b^2}}\right]-2\,i\,\text{ArcTanh}\left[\frac{(a-i\,b)\,\text{Tan}\left[\frac{1}{4}(\pi+2\,i\,x)\right]}{\sqrt{-a^2-b^2}}\right]\right)$$

$$\text{Log}\left[\frac{\left(\frac{1}{2}-\frac{i}{2}\right)\sqrt{-a^2-b^2}\;e^{-x/2}}{\sqrt{-i\,b}\;\sqrt{a+b\,\text{Sinh}[x]}}\right]+$$

$$\left(\text{ArcCos}\left[-\frac{i\,a}{b}\right]+2\,i\left(\text{ArcTanh}\left[\frac{(a+i\,b)\,\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]}{\sqrt{-a^2-b^2}}\right]+\text{ArcTanh}\left[\frac{(a-i\,b)\,\text{Tan}\left[\frac{1}{4}(\pi+2\,i\,x)\right]}{\sqrt{-a^2-b^2}}\right]\right)\right)$$

$$\text{Log}\left[\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{-a^2-b^2}\;e^{x/2}}{\sqrt{-i\,b}\;\sqrt{a+b\,\text{Sinh}[x]}}\right]+$$

$$i\left(\text{PolyLog}\left[2,\left(\left(i\,a+\sqrt{-a^2-b^2}\right)\left(i\,a+b-i\sqrt{-a^2-b^2}\;\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]\right)\right)\Big/\right.\right.$$

$$\left.\left(b\left(i\,a+b+i\sqrt{-a^2-b^2}\;\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]\right)\right)\right]-\text{PolyLog}\Big[2,\left(\left(a+i\sqrt{-a^2-b^2}\right)\right.$$

$$\left.\left.\left.\left(-a+i\,b+\sqrt{-a^2-b^2}\;\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]\right)\right)\Big/\left(b\left(i\,a+b+i\sqrt{-a^2-b^2}\;\text{Cot}\left[\frac{1}{4}(\pi+2\,i\,x)\right]\right)\right)\right]\right)\right)$$

$$\int\frac{\mathbf{x^2}}{\mathbf{a+b\,Sinh[x]}}\,\mathbf{dx}$$

$$\frac{x^2\,\text{Log}\left[1+\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{x^2\,\text{Log}\left[1+\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{2\,x\,\text{PolyLog}\left[2,\frac{b\,e^x}{-a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-$$

$$\frac{2\,x\,\text{PolyLog}\left[2,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{2\,\text{PolyLog}\left[3,\frac{b\,e^x}{-a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{2\,\text{PolyLog}\left[3,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

$$\int\frac{\mathbf{x^3}}{\mathbf{a+b\,Sinh[x]}}\,\mathbf{dx}$$

$$\frac{x^3\,\text{Log}\left[1+\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{x^3\,\text{Log}\left[1+\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{3\,x^2\,\text{PolyLog}\left[2,\frac{b\,e^x}{-a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{3\,x^2\,\text{PolyLog}\left[2,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-$$

$$\frac{6\,x\,\text{PolyLog}\left[3,\frac{b\,e^x}{-a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{6\,x\,\text{PolyLog}\left[3,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{6\,\text{PolyLog}\left[4,\frac{b\,e^x}{-a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{6\,\text{PolyLog}\left[4,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

- *Maple* is only able to integrate $\frac{x^m}{a+b\,\text{Sinh}[x]}$ when m is 1:

```
int (x / (a + b * sinh (x)), x);
```

$$\frac{x\,\text{Log}\left[1+\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{x\,\text{Log}\left[1+\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}+\frac{\text{PolyLog}\left[2,-\frac{b\,e^x}{a-\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}-\frac{\text{PolyLog}\left[2,-\frac{b\,e^x}{a+\sqrt{a^2+b^2}}\right]}{\sqrt{a^2+b^2}}$$

```
int (x^2 / (a + b * sinh (x)), x);
```

$$\int\frac{x^2}{a+b\,\sinh(x)}\,dx$$

```
int (x^3 / (a + b * sinh (x)), x);
```

$$\int\frac{x^3}{a+b\,\sinh(x)}\,dx$$

Note that these systems give similar results to the above for the hyperbolic cosine function.

Hyperbolic Function Integration Problem 3

$$\int \text{Sech}[a + b\,x]^4\,\text{Tanh}[a + b\,x]^n\,dx$$

- *Rubi* maintains the symmetry between the trig and hyperbolic functions:

```
Int[Sec[a + b x]^4 Tan[a + b x]^n, x]
```

$$\frac{\text{Tan}[a + b\,x]^{1+n}}{b\,(1+n)} + \frac{\text{Tan}[a + b\,x]^{3+n}}{b\,(3+n)}$$

```
Int[Sech[a + b x]^4 Tanh[a + b x]^n, x]
```

$$\frac{\text{Tanh}[a + b\,x]^{1+n}}{b\,(1+n)} - \frac{\text{Tanh}[a + b\,x]^{3+n}}{b\,(3+n)}$$

- *Mathematica* is able to integrate the trig expression and the corresponding hyperbolic one:

$$\int \text{Sec}[a + b\,x]^4\,\text{Tan}[a + b\,x]^n\,dx$$

$$\frac{(2 + n + \text{Cos}[2\,(a + b\,x)])\,\text{Sec}[a + b\,x]^2\,\text{Tan}[a + b\,x]^{1+n}}{b\,(1+n)\,(3+n)}$$

$$\int \text{Sech}[a + b\,x]^4\,\text{Tanh}[a + b\,x]^n\,dx$$

$$\left((2 + n + \text{Cosh}[2\,(a + b\,x)])\,\text{Sech}[a + b\,x]^2\,\text{Tanh}[a + b\,x]^{1+n}\right) \Big/ (b\,(1+n)\,(3+n))$$

- *Maple* is unable to integrate the trig expression and returns a huge result for the hyperbolic one:

```
int(sec(a+b*x)^4*tan(a+b*x)^n, x);
```

$$\int \text{Sec}[a + b\,x]^4\,\text{Tan}[a + b\,x]^n\,dx$$

```
int(sech(a+b*x)^4*tanh(a+b*x)^n, x);
```

```
2*(-3*exp(2*a+2*b*x)+exp(6*a+6*b*x)+3*exp(4*a+4*b*x)-1+2*exp(4*a+4*b*x)*n-
     2*n*exp(2*a+2*b*x))/(1+n)/b/(3+n)/(exp(2*a+2*b*x)+1)^3*exp
 (-1/2*n*(-2*ln(exp(a+b*x)-1)-2*ln(1+exp(a+b*x))+2*ln(exp(2*a+2*b*x)+1)+
     I*Pi*csgn(I*(1+exp(a+b*x))/(exp(2*a+2*b*x)+1))^3-
     I*Pi*csgn(I*(1+exp(a+b*x))/(exp(2*a+2*b*x)+1))^2*csgn(I/(exp(2*a+2*b*x)+1))-
     I*Pi*csgn(I*(1+exp(a+b*x))/(exp(2*a+2*b*x)+1))^2*csgn(I*(1+exp(a+b*x)))+
     I*Pi*csgn(I*(1+exp(a+b*x))/(exp(2*a+2*b*x)+1))*
      csgn(I*(1+exp(a+b*x)))*csgn(I/(exp(2*a+2*b*x)+1))+
     I*Pi*csgn(I*(exp(a+b*x)-1)/(exp(2*a+2*b*x)+1)*(1+exp(a+b*x)))^3-
     I*Pi*csgn(I*(exp(a+b*x)-1)/(exp(2*a+2*b*x)+1)*(1+exp(a+b*x)))^2*
      csgn(I*(1+exp(a+b*x))/(exp(2*a+2*b*x)+1))-
     I*Pi*csgn(I*(exp(a+b*x)-1)/(exp(2*a+2*b*x)+1)*(1+exp(a+b*x)))^2*
      csgn(I*(exp(a+b*x)-1))+
     I*Pi*csgn(I*(exp(a+b*x)-1)/(exp(2*a+2*b*x)+1)*(1+exp(a+b*x)))*
      csgn(I*(exp(a+b*x)-1))*csgn(I*(1+exp(a+b*x))/(exp(2*a+2*b*x)+1))))
```

Hyperbolic Function Integration Problem 4

$$\int \text{Sinh}[x]\ \text{Sech}[n\,x]\,dx$$

- The *Rubi* results are simple, expressed in hyperbolic form and grow modestly with n:

```
Int[Sinh[x] Sech[x], x]
```

Log[Cosh[x]]

```
Int[Sinh[x] Sech[2 x], x]
```

$$-\frac{\text{ArcTanh}\left[\sqrt{2}\ \text{Cosh}[x]\right]}{\sqrt{2}}$$

```
Int[Sinh[x] Sech[3 x], x]
```

$$\frac{1}{3}\ \text{ArcTanh}\left[1-\frac{8\,\text{Cosh}[x]^2}{3}\right]$$

```
Int[Sinh[x] Sech[4 x], x]
```

$$\frac{1}{4}\sqrt{2+\sqrt{2}}\ \text{ArcTanh}\left[\frac{2\,\text{Cosh}[x]}{\sqrt{2-\sqrt{2}}}\right]-\frac{1}{4}\sqrt{2-\sqrt{2}}\ \text{ArcTanh}\left[\frac{2\,\text{Cosh}[x]}{\sqrt{2+\sqrt{2}}}\right]$$

- The *Mathematica* results grow unpredictably and uses RootSum when n is 4:

$$\int \text{Sinh}[x]\ \text{Sech}[x]\,dx$$

Log[Cosh[x]]

$$\int \text{Sinh}[x]\ \text{Sech}[2\,x]\,dx$$

$$\frac{1}{4\sqrt{2}}\left(-2\,i\,\text{ArcTan}\left[\frac{\text{Cosh}\left[\frac{x}{2}\right]+\text{Sinh}\left[\frac{x}{2}\right]}{\left(1+\sqrt{2}\right)\text{Cosh}\left[\frac{x}{2}\right]-\left(-1+\sqrt{2}\right)\text{Sinh}\left[\frac{x}{2}\right]}\right]+2\,i\,\text{ArcTan}\left[\frac{\text{Cosh}\left[\frac{x}{2}\right]+\text{Sinh}\left[\frac{x}{2}\right]}{\left(-1+\sqrt{2}\right)\text{Cosh}\left[\frac{x}{2}\right]-\left(1+\sqrt{2}\right)\text{Sinh}\left[\frac{x}{2}\right]}\right]-\right.$$
$$\left.4\,\text{ArcTanh}\left[\sqrt{2}-i\,\text{Tanh}\left[\frac{x}{2}\right]\right]-\text{Log}\left[2\left(\sqrt{2}+2\,\text{Cosh}[x]\right)\right]+\text{Log}\left[-2\sqrt{2}+4\,\text{Cosh}[x]\right]\right)$$

$$\int \text{Sinh}[x]\ \text{Sech}[3\,x]\,dx$$

$$-\frac{1}{3}\,\text{Log}[\text{Cosh}[x]]+\frac{1}{6}\,\text{Log}[-1+2\,\text{Cosh}[2\,x]]$$

$$\int \text{Sinh}[x]\ \text{Sech}[4\,x]\,dx$$

$$\frac{1}{16}\,\text{RootSum}\left[1+\#1^8\,\&,\ \frac{1}{\#1^5}\left(-x-2\,\text{Log}\left[-\text{Cosh}\left[\frac{x}{2}\right]-\text{Sinh}\left[\frac{x}{2}\right]+\text{Cosh}\left[\frac{x}{2}\right]\#1-\text{Sinh}\left[\frac{x}{2}\right]\#1\right]+\right.\right.$$
$$\left.\left.x\,\#1^2+2\,\text{Log}\left[-\text{Cosh}\left[\frac{x}{2}\right]-\text{Sinh}\left[\frac{x}{2}\right]+\text{Cosh}\left[\frac{x}{2}\right]\#1-\text{Sinh}\left[\frac{x}{2}\right]\#1\right]\#1^2\right)\&\right]$$

- The *Maple* results are simple, but expressed in exponential form and not in closed-form when n is 4:

```
int(sinh(x)*sech(x), x);
```

Log[Cosh[x]]

```
int(sinh(x)*sech(2*x), x);
```

$$\frac{\text{Log}\left[1-\sqrt{2}\,e^{x}+e^{2x}\right]}{2\sqrt{2}}-\frac{\text{Log}\left[1+\sqrt{2}\,e^{x}+e^{2x}\right]}{2\sqrt{2}}$$

```
int(sinh(x)*sech(3*x), x);
```

$$-\frac{1}{3}\text{Log}\left[1+e^{2x}\right]+\frac{1}{6}\text{Log}\left[1-e^{2x}+e^{4x}\right]$$

```
int(sinh(x)*sech(4*x), x);
```

2 * sum (_R * ln (exp (2 * x) + (4096 * _R^3 − 48 * _R) * exp (x) + 1), _R = RootOf (32 768 * _Z^4 − 512 * _Z^2 + 1))

Note that these systems give similar results to the above for the hyperbolic cosine function.

Hyperbolic Function Integration Problem 5

$$\int\sqrt{a+b\,\mathrm{Tanh}[x]}\,dx \quad \& \quad \int\sqrt{a+b\,\mathrm{Coth}[x]}\,dx$$

- The *Rubi* results are simple and symmetric:

```
{Int[√(1 + Tanh[x]), x], Int[√(1 + Coth[x]), x]}
```

$$\left\{\sqrt{2}\ \mathrm{ArcTanh}\left[\frac{\sqrt{1+\mathrm{Tanh}[x]}}{\sqrt{2}}\right],\ \sqrt{2}\ \mathrm{ArcCoth}\left[\frac{\sqrt{1+\mathrm{Coth}[x]}}{\sqrt{2}}\right]\right\}$$

```
{Int[√(a + b Tanh[x]), x], Int[√(a + b Coth[x]), x]}
```

$$\left\{-\sqrt{a-b}\ \mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Tanh}[x]}}{\sqrt{a-b}}\right]+\sqrt{a+b}\ \mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Tanh}[x]}}{\sqrt{a+b}}\right],\right.$$
$$\left.-\sqrt{a-b}\ \mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Coth}[x]}}{\sqrt{a-b}}\right]+\sqrt{a+b}\ \mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Coth}[x]}}{\sqrt{a+b}}\right]\right\}$$

- The *Mathematica* results are more complicated involving the imaginary unit and not symmetric:

$$\left\{\int\sqrt{1+\mathrm{Tanh}[x]}\,dx,\ \int\sqrt{1+\mathrm{Coth}[x]}\,dx\right\}$$

$$\left\{\sqrt{2}\ \mathrm{ArcTanh}\left[\frac{\sqrt{1+\mathrm{Tanh}[x]}}{\sqrt{2}}\right],\right.$$
$$\left.\left((1+i)\ \mathrm{ArcTan}\left[\left(\frac{1}{2}+\frac{i}{2}\right)\sqrt{i\,(1+\mathrm{Coth}[x])}\right](1+\mathrm{Coth}[x])^{3/2}\right)\Big/\left(i\,(1+\mathrm{Coth}[x])\right)^{3/2}\right\}$$

$$\left\{\int\sqrt{a+b\,\mathrm{Tanh}[x]}\,dx,\ \int\sqrt{a+b\,\mathrm{Coth}[x]}\,dx\right\}$$

$$\left\{-\sqrt{a-b}\ \mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Tanh}[x]}}{\sqrt{a-b}}\right]+\sqrt{a+b}\ \mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Tanh}[x]}}{\sqrt{a+b}}\right],\right.$$
$$\left(\left(-\sqrt{i\,(a-b)}\ \mathrm{ArcTanh}\left[\frac{\sqrt{i\,(a+b\,\mathrm{Coth}[x])}}{\sqrt{i\,(a-b)}}\right]+\sqrt{i\,(a+b)}\ \mathrm{ArcTanh}\left[\frac{\sqrt{i\,(a+b\,\mathrm{Coth}[x])}}{\sqrt{i\,(a+b)}}\right]\right)\sqrt{a+b\,\mathrm{Coth}[x]}\right)\Big/$$
$$\left.\left(\sqrt{i\,(a+b\,\mathrm{Coth}[x])}\right)\right\}$$

- The *Maple* results are simple and symmetric:

```
[int(sqrt(1+tanh(x)), x), int(sqrt(1+coth(x)), x)];
```

$$\left\{\sqrt{2}\ \mathrm{ArcTanh}\left[\frac{\sqrt{1+\mathrm{Tanh}[x]}}{\sqrt{2}}\right],\ \sqrt{2}\ \mathrm{ArcCoth}\left[\frac{\sqrt{1+\mathrm{Coth}[x]}}{\sqrt{2}}\right]\right\}$$

```
[int(sqrt(a+b*tanh(x)), x), int(sqrt(a+b*coth(x)), x)];
```

$$\left\{-\sqrt{a-b}\ \mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Tanh}[x]}}{\sqrt{a-b}}\right]+\sqrt{a+b}\ \mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Tanh}[x]}}{\sqrt{a+b}}\right],\right.$$
$$\left.-\sqrt{a-b}\ \mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Coth}[x]}}{\sqrt{a-b}}\right]+\sqrt{a+b}\ \mathrm{ArcTanh}\left[\frac{\sqrt{a+b\,\mathrm{Coth}[x]}}{\sqrt{a+b}}\right]\right\}$$

Hyperbolic Function Integration Problem 6

$$\int \frac{\text{Sech}[x]^2}{\sqrt{a - b\,\text{Tanh}[x]^2}}\,dx$$

- The *Rubi* results are simple for both symbolic and numeric a and b:

```
Int[Sech[x]^2/Sqrt[a - b Tanh[x]^2], x]
```

$$-\frac{\text{ArcTan}\left[\frac{\text{Coth}[x]\sqrt{a-b\,\text{Tanh}[x]^2}}{\sqrt{b}}\right]}{\sqrt{b}}$$

```
Int[Sech[x]^2/Sqrt[1 - b Tanh[x]^2], x]
```

$$\frac{\text{ArcSin}\left[\sqrt{b}\,\text{Tanh}[x]\right]}{\sqrt{b}}$$

```
Int[Sech[x]^2/Sqrt[1 - 4 Tanh[x]^2], x]
```

$$\frac{1}{2}\,\text{ArcSin}[2\,\text{Tanh}[x]]$$

- *Mathematica* results are sometimes more complicated depending on a and b:

$$\int \frac{\text{Sech}[x]^2}{\sqrt{a - b\,\text{Tanh}[x]^2}}\,dx$$

$$\left(\text{ArcTan}\left[\frac{\sqrt{2}\,\sqrt{b}\,\text{Sinh}[x]}{\sqrt{a+b+(a-b)\,\text{Cosh}[2\,x]}}\right]\sqrt{a+b+(a-b)\,\text{Cosh}[2\,x]}\;\text{Sech}[x]\right)\Big/\left(\sqrt{2}\,\sqrt{b}\,\sqrt{a-b\,\text{Tanh}[x]^2}\right)$$

$$\int \frac{\text{Sech}[x]^2}{\sqrt{1 - b\,\text{Tanh}[x]^2}}\,dx$$

$$\frac{\text{ArcSin}\left[\sqrt{b}\,\text{Tanh}[x]\right]}{\sqrt{b}}$$

$$\int \frac{\text{Sech}[x]^2}{\sqrt{1 - 4\,\text{Tanh}[x]^2}}\,dx$$

$$\frac{\text{ArcTanh}\left[\frac{2\sqrt{2}\,\text{Sinh}[x]}{\sqrt{-5+3\,\text{Cosh}[2\,x]}}\right]\sqrt{-5+3\,\text{Cosh}[2\,x]}\;\text{Sech}[x]}{2\sqrt{2-8\,\text{Tanh}[x]^2}}$$

- *Maple* is unable to integrate $\frac{\text{Sech}[x]^2}{\sqrt{a-b\,\text{Tanh}[x]^2}}$ for symbolic and numeric variables a and b:

```
int(sech(x)^2/sqrt(a-b*tanh(x)^2),x);
```

$$\int \frac{\text{Sech}[x]^2}{\sqrt{a - b\,\text{Tanh}[x]^2}}\,dx$$

```
int (sech (x) ^2 / sqrt (1 - b * tanh (x) ^2), x);
```

$$\int \frac{\text{Sech}[x]^2}{\sqrt{1 - b\,\text{Tanh}[x]^2}}\,dx$$

```
int (sech (x) ^2 / sqrt (1 - 4 * tanh (x) ^2), x);
```

$$\int \frac{\text{Sech}[x]^2}{\sqrt{1 - 4\,\text{Tanh}[x]^2}}\,dx$$

Hyperbolic Function Integration Problem 7

$$\int \frac{\mathrm{Tanh}[x]}{\sqrt{a + b\,\mathrm{Tanh}[x]^4}}\, dx$$

- *Rubi* is able to integrate the expression:

```
Int[Tanh[x]/Sqrt[a + b Tanh[x]^4], x]
```

$$\frac{\mathrm{ArcTanh}\left[\frac{\sqrt{a+b}\,\sqrt{a+b\,\mathrm{Tanh}[x]^4}}{a+b\,\mathrm{Tanh}[x]^2}\right]}{2\sqrt{a+b}}$$

- *Mathematica* is unable to integrate the expression:

$$\int \frac{\mathrm{Tanh}[x]}{\sqrt{a + b\,\mathrm{Tanh}[x]^4}}\, dx$$

$$\int \frac{\mathrm{Tanh}[x]}{\sqrt{a + b\,\mathrm{Tanh}[x]^4}}\, dx$$

- *Maple* is able to integrate the expression:

```
int(tanh(x)/sqrt(a+b*tanh(x)^4),x);
```

$$\frac{\mathrm{ArcTanh}\left[\frac{a+b\,\mathrm{Tanh}[x]^2}{\sqrt{a+b}\,\sqrt{a+b\,\mathrm{Tanh}[x]^4}}\right]}{2\sqrt{a+b}}$$

Hyperbolic Function Integration Problem 8

$$\int \frac{\sqrt{\text{Sinh}[a+b\,x]}}{\sqrt{\text{Cosh}[a+b\,x]}}\,dx$$

- The *Rubi* results are symmetric and involve only elementary functions:

$$\text{Int}\left[\sqrt{\frac{\text{Sinh}[a+b\,x]}{\text{Cosh}[a+b\,x]}},\,x\right]$$

$$-\frac{\text{ArcTan}\left[\sqrt{\text{Tanh}[a+b\,x]}\right]}{b}+\frac{\text{ArcTanh}\left[\sqrt{\text{Tanh}[a+b\,x]}\right]}{b}$$

$$\text{Int}\left[\frac{\sqrt{\text{Sinh}[a+b\,x]}}{\sqrt{\text{Cosh}[a+b\,x]}},\,x\right]$$

$$-\frac{\text{ArcTan}\left[\frac{\sqrt{\text{Sinh}[a+b\,x]}}{\sqrt{\text{Cosh}[a+b\,x]}}\right]}{b}+\frac{\text{ArcTanh}\left[\frac{\sqrt{\text{Sinh}[a+b\,x]}}{\sqrt{\text{Cosh}[a+b\,x]}}\right]}{b}$$

- The *Mathematica* results are not symmetric and involve a hypergeometric function:

$$\int\sqrt{\frac{\text{Sinh}[a+b\,x]}{\text{Cosh}[a+b\,x]}}\,dx$$

$$-\frac{\text{ArcTan}\left[\sqrt{\text{Tanh}[a+b\,x]}\right]}{b}-\frac{\text{Log}\left[-1+\sqrt{\text{Tanh}[a+b\,x]}\right]}{2\,b}+\frac{\text{Log}\left[1+\sqrt{\text{Tanh}[a+b\,x]}\right]}{2\,b}$$

$$\int\frac{\sqrt{\text{Sinh}[a+b\,x]}}{\sqrt{\text{Cosh}[a+b\,x]}}\,dx$$

$$-\frac{2\,\sqrt{\text{Cosh}[a+b\,x]}\;\text{Hypergeometric2F1}\left[\frac{1}{4},\,\frac{1}{4},\,\frac{5}{4},\,\text{Cosh}[a+b\,x]^2\right]\,\text{Sinh}[a+b\,x]^{3/2}}{b\,\left(-\text{Sinh}[a+b\,x]^2\right)^{3/4}}$$

- *Maple* is able to integrate $\sqrt{\text{Tanh}[a+b\,x]}$ but not $\sqrt{\text{Sinh}[a+b\,x]\,/\,\text{Cosh}[a+b\,x]}$:

```
int(sqrt(tanh(a+b*x)), x);
```

$$-\frac{\text{ArcTan}\left[\sqrt{\text{Tanh}[a+b\,x]}\right]}{b}-\frac{\text{Log}\left[-1+\sqrt{\text{Tanh}[a+b\,x]}\right]}{2\,b}+\frac{\text{Log}\left[1+\sqrt{\text{Tanh}[a+b\,x]}\right]}{2\,b}$$

```
int(sqrt(sinh(a+b*x)/cosh(a+b*x)), x);
```

$$\int\sqrt{\frac{\text{Sinh}[a+b\,x]}{\text{Cosh}[a+b\,x]}}\,dx$$

- The *Maple* result involves complex exponentials and an elliptic integral function:

```
int(sqrt(sinh(a+b*x))/sqrt(cosh(a+b*x)), x);
```

```
2/3/b*exp(a+b*x)*(exp(a+b*x)^2+1)/((exp(a+b*x)^2+1)*exp(a+b*x))^(1/2) -
 4/3*I/b*(-I*(I+exp(a+b*x)))^(1/2)*2^(1/2)*(I*(-I+exp(a+b*x)))^(1/2)*
  (I*exp(a+b*x))^(1/2)/(exp(a+b*x)+exp(a+b*x)^3)^(1/2)*
  EllipticF((-I*(I+exp(a+b*x)))^(1/2), 1/2*2^(1/2))
```