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Transient oscillatory patterns in the diffusive non-local blowfly equation with delay under the zero-flux boundary condition

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Abstract

In this paper, we study the spatial-temporal patterns of the solutions to the diffusive non-local Nicholson’s blowflies equations with time delay (maturation time) subject to the no flux boundary condition. We establish the existence of both spatially homogeneous periodic solutions and various spatially inhomogeneous periodic solutions by investigating the Hopf bifurcations at the spatially homogeneous steady state. We also compute the normal form on the centre manifold, by which the bifurcation direction and stability of the bifurcated periodic solutions can be determined. The results show that the bifurcated homogeneous periodic solutions are stable, while the bifurcated inhomogeneous periodic solutions can only be stable on the corresponding centre manifold, implying that generically the model can only allow transient oscillatory patterns. Finally, we present some numerical simulations to demonstrate the theoretic results. For these transient patterns, we derive approximation formulas which are confirmed by numerical simulations.

Mathematics Subject Classification: 35B32, 35K57, 92B05

(Some figures may appear in colour only in the online journal)
1. Introduction

Gurney et al [5] proposed the time delayed ODE model
\[
\frac{dw(t)}{dt} = -dw(t) + pw(t-\tau) e^{-qw(t-\tau)}
\] (1.1)
to describe the population dynamics of blowflies, hoping to explain the oscillatory phenomena in Nicholson’s laboratory experiments [17]. Here \(w(t)\) is the size of the mature blowfly population at time \(t\), \(p\) is the maximum per capita daily egg production rate, \(1/q\) is the size at which the blowfly population reproduces at its maximum rate, \(d\) is the per capita daily adult death rate and \(\tau\) is the maturation time. Since [5], (1.1) has been widely quoted as the Nicholson blowflies equation and has been extensively studied in the literature (see, e.g., [1, 10–12, 23, 28] and references therein).

In fact, (1.1) can be derived from the following age-structured population model (see, e.g., [19])
\[
\begin{cases}
\frac{\partial u(t, a)}{\partial t} + \frac{\partial u(t, a)}{\partial a} = -d(a)u(t, a), & t > 0, a > 0, \\
u(t, 0) = b(w(t)), & t > 0,
\end{cases}
\] (1.2)
where \(u(t, a)\) is the population density of age \(a\) at time \(t\), \(d(a)\) is the age-dependent death rate, \(w(t) = \int_{\tau}^{\infty} u(t, a) \, da\) is the total density of the mature population at time \(t\), \(\tau\) is the maturation time and \(b(w)\) is the Ricker type’s birth function: \(b(w) = pw e^{-qw}\).

Taking into account spatial diffusion of the population in a one dimensional domain \(\Omega \subset \mathbb{R}\), parallel to (1.2), one can obtain a general diffusive age-structured model given by (see, e.g., [15])
\[
\begin{cases}
\frac{\partial u(t, a, x)}{\partial t} + \frac{\partial u(t, a, x)}{\partial a} = -d(a)u(t, a, x) + \epsilon \int_{\Omega} K_\alpha(x, y) b(w(t-\tau, y)) \, dy, & t > 0, a > 0, x \in \Omega \subset \mathbb{R}, \\
u(t, 0, x) = b(w(t, x)), & t > 0, x \in \Omega \subset \mathbb{R},
\end{cases}
\] (1.3)
where \(u(t, a, x)\) now is the population density of age \(a\) at time \(t\) and location \(x\), \(D(a)\) is the age-dependent diffusion rate, \(w(t, x) = \int_{\tau}^{\infty} u(t, a, x) \, da\) is the total density of the mature population at time \(t\) and location \(x\). For fundamental theory and many interesting topics on age-structured models, we refer to [16, 26, 27].

Under the assumption that the diffusion rate and death rate of the mature population are age-independent, that is,
\[
D(a) = D_m, \quad d(a) = d \quad \text{for all } a \geq \tau,
\]
one can derive the following equation from (1.3) for the mature population \(w(t, x)\):
\[
\frac{\partial w(t, x)}{\partial t} = D_m \frac{\partial^2 w(t, x)}{\partial x^2} - dw(t, x) + \epsilon \int_{\Omega} K_\alpha(x, y) b(w(t-\tau, y)) \, dy, \quad t > 0, \quad x \in \Omega,
\] (1.4)
where \(\alpha = \int_0^\tau D(a) \, da\) measures the mobility of the immature population and \(\epsilon = \exp(-\int_0^\tau d(a) \, da)\) is the survival factor accounting for the proportion of individuals that can survive the immature period, while the kernel function \(K_\alpha(x, y)\) depends on the boundary condition, accounting for the probability that an individual born in location \(y\) will have moved to location \(x\) after \(\tau\) units of time (maturation time). Thus, the last term on the right hand side of (1.4) sums up all individuals born in the domain \(\tau\) time units ago who have moved to location \(x\) at maturation. There is one thing in common for the kernel function \(K_\alpha(x, y)\) under
Various boundary conditions, that is, as \( \alpha \to 0 \), \( K_\alpha(x, y) \) tends to the Dirac delta function of \( x - y \): 
\[
K_\alpha(x, y) \to \delta(x - y),
\]
reducing (1.4) to the spatially local equation:
\[
\frac{\partial w(t, x)}{\partial t} = D_m \frac{\partial^2 w(t, x)}{\partial x^2} - d w(t, x) + \varepsilon b(w(t - \tau, x)), \quad t > 0, \quad x \in \Omega.
\]

For \( \Omega = \mathbb{R} \), So et al [21] showed that \( K_\alpha \) is nothing but the heat kernel function with the parameter \( \alpha \) determining its flatness. There have been many works on (1.4) or some special cases of (1.4) (including (1.5)), dealing with such interesting topics as stability of the constant steady states, travelling wave fronts connecting two constant steady states and the stability of these fronts, as well as the asymptotic speed of spread. Such an unbounded domain case is not the concern of this paper, and hence, we will not go further along this line. An interested reader is referred to, for example, [2, 14, 21, 24, 36] and the references therein.

For \( \Omega = [0, L] \), Liang et al [13] obtained the explicit forms of \( K_\alpha(x, y) \) under some common boundary conditions (including Neumann, Dirichlet, Robin and periodic conditions) at the two ends \( x = 0, L \) and explored numerical methods for the solutions to the resulting non-local reaction–diffusion equations. For the special case (1.5) (i.e., \( \alpha \to 0 \)), under zero Dirichlet boundary condition (a scenario for hostile boundary), So and Yang [22] investigated the global stability of the steady states of (1.5), So et al [20] numerically explored Hopf bifurcation of (1.5), and Su et al [25] analysed the existence and nonexistence of the positive steady state of (1.5); while under zero-flux boundary condition, Yang and So [31] studied the stability of the steady states and the existence of Hopf bifurcation, Yi et al [32] and Yi and Zou [34, 35] identified some ranges of the parameters within which the delay \( \tau \) has no impact on the global dynamics of (1.5). For the true non-local case, Xu and Zhao [30], Zhao [38] and Yi and Zou [37] have obtained some results on the threshold dynamics of (1.4) under zero Dirichlet/Neumann boundary condition which support convergence of solutions to steady states.

In this paper, we consider the true non-local equation (1.4) with the Ricker type birth function \( b(w) = \frac{p}{d} w e^{-q w} \) on the domain \( \Omega = [0, \pi] \) subject to the zero-flux boundary condition. By the results in [30, 34, 35, 38], it is known that when \( 1 < \frac{p \varepsilon}{d} < e^2 \), the equation has a positive constant steady state \( E^* \left( \frac{p \varepsilon}{d} \right) \) which attracts all positive solutions to this boundary value problem, and therefore there will be no temporal and/or spatial patterns arising from (1.4). It is natural to ask what will happen when \( \frac{p \varepsilon}{d} > e^2 \), and addressing this question constitutes the goal of this paper.

To proceed, we note that for \( \Omega = [0, \pi] \) and with the zero-flux boundary condition, [13] has shown that the kernel function \( K_\alpha(x, y) \) in (1.4) is
\[
K_\alpha(x, y) = \frac{1}{\pi} \left[ 1 + \sum_{n=1}^{\infty} \left( \cos n(x + y) + \cos n(x - y) \right) e^{-\alpha n^2} \right].
\]

Plugging this into (1.4), re-scaling by
\[
\hat{w}(t) = q w \left( \frac{t}{\tau} \right), \quad \hat{D}_m = \tau D_m, \quad \hat{\tau} = d \tau, \quad \text{and} \quad \beta = \frac{p \varepsilon}{d}
\]
and dropping the hats for the simplicity of notations, we obtain
\[
\begin{cases}
\frac{\partial w(t, x)}{\partial t} = \frac{\partial^2 w(t, x)}{\partial x^2} - \tau w(t, x) + \frac{\beta \tau}{\pi} \int_0^\pi w(t - 1, y) \exp[-w(t - 1, y)] \left[ 1 + \sum_{n=1}^{\infty} \left( \cos n(x + y) + \cos n(x - y) \right) e^{-\alpha n^2} \right] dy, \quad x \in [0, \pi], \\
\frac{\partial w(t, 0)}{\partial x} = \frac{\partial w(t, \pi)}{\partial x} = 0.
\end{cases}
\]

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As such, in the rest of this paper we only need to focus on the boundary value problem (1.6) in the range $\beta \in (e^2, \infty)$.

Section 2 is devoted to a thorough Hopf bifurcation analysis for (1.6), using $\beta \in (e^2, \infty)$ as a bifurcation parameter. We show that there is a sequence of critical values for $\beta$ at which Hopf bifurcations occur. Among these critical values, the first one presents Hopf bifurcation generating spatially homogeneous periodic solutions around the spatially homogeneous positive steady state $E^+ (\beta)$, while the rest give rise to periodic solutions that are spatially inhomogeneous, demonstrating various spatial patterns. By applying the centre manifold theory and the normal form method, we are also able to provide, in the appendix, an explicit algorithm for determining the direction of the Hopf bifurcation and stability of the bifurcating periodic solutions. We also study the dependence of Hopf bifurcation points and the bifurcated oscillations on some model parameters including the diffusion rates of the mature and immature populations. We prove that there exist spatially heterogeneous periodic solutions which are $\cos(nx)$-perturbations of $E^+ (\beta)$; and show that for some parameter values, they can be stable on the centre manifold but unstable in the whole phase space. Therefore, generically (1.6) only allows transient spatial patterns. In section 3, we present numerical simulations which demonstrate our theoretical results; in particular, the simulations show that a solution with a $\cos(nx)$-like initial function tends to a $\cos(nx)$-like time-periodic solution in a relatively long time, and then it eventually converges to a spatially homogeneous periodic solution. The numerical simulations shows that the non-locality caused by the mobility of the immature population (measured by $\alpha$) will shorten the duration of such transient spatial patterns, and to our best knowledge, this is the first time that such an effect is observed/reported.

To end this introduction, we point out that Gourley and Ruan [4] generalized (1.5) to the following equation by introducing distributed delay

$$\frac{\partial w(t, x)}{\partial t} = D_m \frac{\partial^2 w(t, x)}{\partial x^2} - dw(t, x) + \varepsilon b \left( \int_{-\infty}^{0} f(s) w(t+s, x) \, ds \right).$$

(1.7)

Hu and Yuan [8] further generalized (1.7) to a spatially non-local version

$$\frac{\partial w(t, x)}{\partial t} = D_m \frac{\partial^2 w(t, x)}{\partial x^2} - dw(t, x) + \varepsilon b \left( \int_{-\infty}^{0} \int_{\Omega} f(s, x, y) w(t+s, y) \, dy \, ds \right).$$

(1.8)

In both [4, 8], by using delay as the bifurcation parameter, the authors considered Hopf bifurcations at the positive constant steady state, but only explored spatially homogeneous periodic solutions. Note that for a PDE system subject to zero-flux boundary condition, a bifurcated spatially homogeneous periodic solution can also be bifurcated in the corresponding kinetic equation (equation without diffusion), the effect of diffusion cannot be reflected by such bifurcations. Overall, spatially inhomogeneous time-periodic solutions in reaction–diffusion systems subject to zero-flux boundary condition have been overlooked. The recent work Yi et al [33] is an exception, where for a PDE model without delay, the authors observed that the bifurcated spatially inhomogeneous periodic solution are unstable. But here we have gone further by showing that such a spatially inhomogeneous periodic solution can be stable in the corresponding centre manifold, and can be numerically observable in a relatively long time period if the initial distribution is close to the central manifold (being a $\cos(nx)$-like shape). We would also like to point out that the system (1.8) was proposed as a mathematical generalization of (1.7), and hence, of (1.5); our model (1.6) is rigorously derived from the standard age structured PDE model (1.3), and thus, all terms and parameters in (1.6) have clear biological explanations.
2. Characteristic equations and Hopf bifurcations

In this section, we study the local stability of the spatially homogeneous positive steady state and Hopf bifurcations for (1.6). Unlike in [4, 8] where delay was used as the bifurcation parameter, we will use $\beta$ as the bifurcation parameter.

Denote $X := \{ \phi \in W^{2,2}(0, \pi), \phi'(0) = \phi' (\pi) = 0 \}$ and let $C = C([-1,0], X)$ be the Banach space of continuous $X$-values functions on $[-1, 0]$ equipped with the sup norm. From [29] or [7], (1.6) with the following initial condition

\[ w(t, x) = \eta(\theta, x) \in C([-1,0], W^{1,2}(0, \pi)), \quad t \in [-1,0] \]

have a unique local solution. By an easy calculation we know that (1.6) has a spatially homogeneous steady state $E^\ast = \ln \beta$. Since we always assume that $\beta > e^2$ in the rest of this paper, it follows that $E^\ast$ is indeed a positive steady state. The linearization of (1.6) about $E^\ast$ is

\[
\begin{cases}
\frac{\partial w(t, x)}{\partial t} = D_m \frac{\partial^2 w(t, x)}{\partial x^2} - \tau w(t, x) + \frac{(1 - \ln \beta) \tau}{\pi} \int_0^\pi w(t - 1, y) dy, & x \in [0, \pi], \ t > 0 \\
\frac{\partial w(t, 0)}{\partial x} = \frac{\partial w(t, \pi)}{\partial x} = 0, & t > 0.
\end{cases}
\]

(2.1)

It is well known that the eigenvalue problem

\[\phi''(x) + \nu \phi(x), \quad \phi'(0) = \phi'(\pi) = 0,\]

has eigenvalues $\nu = -n^2$, $n \in \mathbb{N}_0$, with the corresponding normalized eigenfunctions $\rho_n \cos(nx)$, where $\rho_0 = \sqrt{1/\pi}$ and $\rho_n = \sqrt{2/\pi}$ for $n > 0$, and $\{\rho_n \cos(nx)\}_{n=0}^{\infty}$ forms a complete and orthonormal basis for $X$. Here and in the sequel, we follow the tradition to denote by $\mathbb{N}$ the set of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

To explore the stability of $E^\ast$, we plug the trial function $w(t, x) = e^{it} \phi(x)$ into (2.1). Before this, we note that $\phi(x)$ can be expressed in terms of $\{\rho_n \cos(nx)\}_{n=0}^{\infty}$ as

\[\phi(x) = \sum_{n=0}^{\infty} c_n \rho_n \cos(nx).\]

Making use of the orthogonality of $\{\rho_n \cos(nx)\}_{n=0}^{\infty}$ after plugging, we are led to

\[c_n (\lambda + \tau + D_m n^2 - \tau (1 - \ln \beta)e^{-\omega n}) = 0, \quad n \in \mathbb{N}_0. \]

(2.2)

Denote $\Delta_n(\lambda) = \lambda + \tau + D_m n^2 - \tau (1 - \ln \beta)e^{-\omega n}$. Note that $\phi(x)$ is non-trivial if and only if $c_n \neq 0$ for some $n \in \mathbb{N}_0$, implying that $\lambda$ is an eigenvalue of the linearization equation (2.1) if and only if there exists $n \in \mathbb{N}_0$ such that $\Delta_n(\lambda) = 0$. Thus, (2.1) has the following set of characteristic equations:

\[\Delta_n(\lambda) = 0, \quad n \in \mathbb{N}_0. \]

(2.3)

It is easy to see that $\lambda = 0$ is not an eigenvalue. If $\lambda = \pm i \omega$ ($\omega > 0$) are the solutions of (2.3), then substituting it into (2.3) and separating the real and imaginary parts, we obtain the following equations

\[
\begin{aligned}
\omega &= \tau (\ln \beta - 1)e^{-\omega n} \sin \omega, \\
\tau + D_m n^2 &= \tau (1 - \ln \beta)e^{-\omega n} \cos \omega, \quad n \in \mathbb{N}_0.
\end{aligned}
\]

(2.4)

For fixed $\tau$, $D_m$ and $\omega$, let $\omega_j = \left(\frac{\pi}{2} + 2j\pi, \pi + 2j\pi\right)$ be the solution of the following equation

\[\tan \omega = -\frac{\omega}{\tau + D_m n^2}, \quad n \in \mathbb{N}_0, \quad j \in \mathbb{N}_0.\]

(2.5)
and
\[
\beta_n^j = \exp \left(1 - \frac{\tau + D_m n^2}{\tau e^{-an^2} \cos \omega_n^*} \right), \quad n \in \mathbb{N}_0, \quad j \in \mathbb{N}_0.
\] (2.6)

From the definitions of \(\omega_n^*\) and \(\beta_n^j\), we know that \(\pm i \omega_n^*\) are the roots of \(\Delta_n(\lambda) = 0\) with \(\beta = \beta_n^j\) and \(\beta_n^j\) can also be expressed as
\[
\beta_n^j = \exp \left(1 + \frac{\omega_n^j}{\tau e^{-an^2} \sin \omega_n^*} \right), \quad n \in \mathbb{N}_0, \quad j \in \mathbb{N}_0.
\] (2.7)

It is obvious that \(\omega_n^j\) is increasing in \(\tau\) and \(D_m\) for \(n \in \mathbb{N}\) and \(j \in \mathbb{N}_0\), and is independent of \(\alpha\), and \(\beta_n^j > e^2\). From the above discussions, we can easily obtain following further information about \(\omega_n^j\) and \(\beta_n^j\).

**Lemma 2.1.** The following statements hold.

(i) \(\pm i \omega_n^j, \quad n \in \mathbb{N}_0, \quad j \in \mathbb{N}_0\) are the purely imaginary roots of (2.3) with \(\beta = \beta_n^j\), and (2.3) has no other purely imaginary root

(ii) For any fixed \(n \in \mathbb{N}_0\), \(\beta_n^j < \beta_n^k\) if \(j < k\). For any fixed \(j \in \mathbb{N}_0\), \(\beta_n^j < \beta_m^j\) if \(n < m\).

Therefore, \(\beta_n^j < \beta_m^j\) if \(n \leq m\), \(j \leq k\) and \((n, j) \neq (m, k)\)

(iii) \(\beta_n^j, \quad j \in \mathbb{N}_0\) is independent on \(D_m\) and \(\alpha\). For any fixed \(n \in \mathbb{N}\) and \(j \in \mathbb{N}_0\), \(\beta_n^j\) is an increasing function of \(D_m\) and \(\alpha\).

Let \(\lambda(\beta) = \gamma(\beta) + i \omega(\beta)\) be the root of (2.3) satisfying \(\gamma(\beta_n^j) = 0\) and \(\omega(\beta_n^j) = \omega_n^j\) when \(\beta\) is close to \(\beta_n^j\), for \(n \in \mathbb{N}_0, \quad k \in \mathbb{N}_0\). Then we have the following transversality result.

**Lemma 2.2.** \(\gamma'(\beta_n^j) > 0\) for any \(n \in \mathbb{N}_0, \quad k \in \mathbb{N}_0\).

**Proof.** Taking the derivative on both side of (2.3) with respect to \(\beta\), and replacing \(\beta\) by \(\beta_n^j\), we have
\[
\frac{d \lambda}{d \beta}(\beta_n^j) = \frac{-\tau e^{-i \omega_n^j} e^{-an^2}}{\beta_n^j [1 + \tau (1 - \ln \beta_n^j) e^{-i \omega_n^j} e^{-an^2}]}.
\] (2.8)

Using the fact that \(\lambda = i \omega_n^j\) is the solution of (2.3), replacing \(\tau (1 - \ln \beta_n^j) e^{-i \omega_n^j} e^{-an^2}\) by \(i \omega_n^j + \tau + D_m n^2\) and matching the real parts of both sides of the resulting equation yields
\[
\gamma'(\beta_n^j) = -\frac{\tau e^{-an^2} \cos \omega_n^j}{\beta_n^j [(1 + \tau + D_m n^2)^3 + (\omega_n^j)^2]} > 0.
\] (2.9)

From lemmas 2.1 and 2.2 and using the result of Ruan and Wei [18, corollary 2.4], we are now in the position to state a conclusion on the distribution of roots of (2.3).

**Lemma 2.3.** The following statements hold.

(i) If \(\beta \in (e^2, \beta_0^j)\) (note that \(\beta_0^j > e^2\)), then all roots of (2.3) have negative real parts.

(ii) For \(n \in \mathbb{N}_0\), (2.3) has purely imaginary roots if and only if \(\beta = \beta_n^j, \quad j \in \mathbb{N}_0\). When \(\beta = \beta_0^j\), all the roots of (2.3), except \(\pm i \omega_n^j\), have negative real parts.

(iii) If \(\beta > \beta_0^j\), then (2.3) has at least one pair of roots with positive real parts.

The following result on the stability of the spatially homogeneous steady state \(E^* = \ln \beta\) follows directly from the above lemma.
Theorem 2.4. For any fixed \( \tau \), the spatially homogeneous steady state \( E^* = \ln \beta \) is asymptotically stable when \( \beta \in (e^2, \beta_0^a) \) and unstable when \( \beta > \beta_0^a \). Moreover, (1.6) undergoes a Hopf bifurcation at \( w = \ln \beta \) and \( \beta = \beta_0^a \).

Remark 2.5. The first Hopf bifurcation value \( \beta_0^a \) is independent of diffusion coefficients \( D_m \) and \( \alpha \), so is the stability of the spatially homogeneous steady state \( E^* \).

In what follows, we always impose the following assumption when we study Hopf bifurcations around \( \beta = \beta_k^n \) for \( k, n \in \mathbb{N}_0 \).

\[ (H_k^n) \quad \text{Equation } \Delta_n(\lambda) = 0 \text{ with } \beta = \beta_k^n \text{ has only one pair of purely imaginary roots } \pm i\omega_k^n. \]

This assumption is equivalent to the following condition due to lemma 2.1:

\[ (H_k^n)^* \beta_k^n \neq \beta_m^n, \quad \forall (m, j) \in \{(m, j) : m > n, j < k\} \cup \{(m, j) : m < n, j > k\}. \]

With the above preparation, we have the following theorem.

Theorem 2.6. If \( (H_k^n)^* \) holds for \( n, k \in \mathbb{N}_0 \), then (1.6) undergoes a Hopf bifurcation at \( w = \ln \beta_k^n \) when \( \beta = \beta_k^n \), and the bifurcating periodic solutions can be written as

\[ w(t, x) = \ln \beta + 2 \sqrt{-\frac{(\beta - \beta_k^n)^{\omega'(\beta_k^n)}(\beta_k^n)}{\text{Re}[C_1^{ak}(0)]}} \rho_n \cos \left( \frac{2\pi t}{T} \right) \cos(nx) + O(\beta - \beta_k^n), \quad (2.10) \]

where

\[ T = \frac{2\pi}{\omega_k^n} \left[ 1 - \tau_{2a} \frac{\gamma'(\beta_k^n)(\beta - \beta_k^n)}{\text{Re}[C_1^{ak}(0)]} + O((\beta - \beta_k^n)^2) \right], \]

\[ \tau_{2a} = -\frac{1}{\omega_k^n} \left[ \text{Im}[C_1^{ak}(0)] - \frac{\text{Re}[C_1^{ak}(0)] \omega'(\beta_k^n)}{\gamma'(\beta_k^n)} \right], \]

and \( C_1^{ak}(0) \) is a constant in the normal form which is calculated in the appendix.

Proof. The existence of Hopf bifurcation at \( \beta = \beta_k^n \) directly follows from lemmas 2.1–2.2 and the expression (2.10), and the formulas for \( T \) and \( \tau_{2a} \) are direct result of applying the centre manifold theorem in [3] and normal form method in [6], with detailed derivation given in the appendix.

Remark 2.7. Based on the above discussions, we also have the following remarks.

(i) If \( (H_k^n)^* \) with \( n \neq 0 \) holds and \( \text{Re}[C_1^{ak}(0)] \neq 0 \), then (1.6) admits spatially inhomogeneous periodic solutions with form (2.10) in a left or right neighbourhood of \( \beta_k^n \). From expression (2.10), we see that at time \( t \) these solutions are \( \cos(nx) \)-perturbations of the spatially homogeneous steady state \( \ln \beta \).

(ii) Using lemma (2.3)-(iii), we know that when \( \beta > \beta_0^a \), an unstable manifold in a small neighbourhood of the steady state always exists. Therefore, all inhomogeneous periodic solutions arising from local Hopf bifurcations are unstable in the phase space. But its stability on the centre manifold is determined by the sign of the corresponding \( \text{Re}[C_1^{ak}(0)] \).

(iii) From lemma (2.1), we see that for any fixed \( n \geq 1 \) the critical value \( \beta_0^n \), from which \( \cos(nx) \)-like time-periodic solutions are bifurcated, is increasing both with respect to the mature diffusion coefficients \( D_m \) and in the immature mobility constant \( \alpha \). Therefore, increasing \( \alpha \) may will enlarge \( \beta_0^n \) and therefore, may help eliminate \( \cos(nx) \)-like oscillations in time around the steady state.
Next we explore in more detail the impact of the mature diffusion rate $D_m$ on the existence of imaginary roots in spatially inhomogeneous periodic solutions. To this end, we use $D_m$ as a parameter. We have seen from lemmas 2.1 and 2.3 that if $\beta \leq \beta_0^j$, then (2.3) has no purely imaginary roots for all $D_m > 0$ and $\alpha \geq 0$. Thus, in the sequel we only need to consider those $\beta > \beta_0^j$ with $\beta \neq \beta_0^j$, $j \in \mathbb{N}$ are critical values for $\beta$ that are independent of $D_m$ at which, only spatially homogeneous periodic solutions are bifurcated. For convenience, we set $M = \{ \beta : \beta > \beta_0^0 \text{ and } \beta \neq \beta_0^j, j \in \mathbb{N} \}$.

Assume that $\lambda = \pm i{\mu}$ are a pair of purely imaginary roots of $\Delta_n(\lambda) = 0$. Then, the real parts in the equation $\Delta_n(i{\mu}) = 0$ lead to

$$K(n){\mu} = \sin(\mu) \quad (2.11)$$

and the imaginary parts result in

$$(\tau + D_m n^2) K(n) = -\cos(\mu) \quad (2.12)$$

where

$$K(n) = \frac{e^{\alpha n^2}}{\tau (\ln \beta - 1)}.$$ 

Note that $K(n) > 0$ (since $\beta \in M$) and $K(n)$ is increasing in $n$ with $K(\infty) = \infty$ if $\alpha > 0$. Thus, if $K(1) \geq 2/\pi$, then (2.11) cannot have any root $\mu > 0$ at which $\cos \mu < 0$ (see figure 1), implying that $\Delta_n(\lambda) = 0$ cannot have purely imaginary roots for all $n \in \mathbb{N}$.

When $K(1) < 2/\pi$, there is an $N \geq 1$ such that $K(n) < 2/\pi$ for $n \leq N$ and $K(n) \geq 2/\pi$ for $n > N$. In such a case, for each $n \leq N$, define

$$I_n = \max \{ j \in \mathbb{N}_0 : (2.11) \text{ has a solution } \mu^j_n \text{ in the interval } (\pi/2 + 2j\pi, \pi + 2j\pi) \}.$$ 

(2.13)

Then, for $0 \leq j < k \leq I_n$, we have

$$-\cos(\mu^j_n) > -\cos(\mu^k_n) \geq -\cos(\mu^k_n),$$

$$-\cos(\mu^j_n) \geq -\cos(\mu^k_n).$$

Note that

$$(\tau + D_m n^2) K(n) = \frac{e^{\alpha n^2}(\tau + D_m n^2)}{\tau (\ln \beta - 1)} > \frac{e^\alpha (\tau + D_m)}{\tau (\ln \beta - 1)} > \frac{e^\alpha}{\ln \beta - 1}.$$ 

Thus, if

$$\frac{e^\alpha}{\ln \beta - 1} \geq -\cos(\mu^j_n) \quad (2.14)$$

Figure 1. Illustration of the solution of the first equation of (2.11)
then none of the $\mu_i^n$, $j = 0, 1, \ldots, I_n$, can satisfy (2.12), implying that (2.3) cannot have purely imaginary roots for $D_m > 0$. On the other hand, if (2.14) is reversed, i.e.,

$$\frac{e^{it}}{\ln \beta - 1} < -\cos(\mu_i^n),$$

then, there exists a $J_n \in \mathbb{N}$ with $J_n < I_n$ such that

$$(D_m)^{J_n} := \left(1 - \ln \beta \cos(\mu_i^n) e^{-at \tau} \right) (n^2)^{-1} > 0.$$ 

In other words, $\mu_i^n$ also satisfies (2.12) when $D_m = (D_m)^{J_n}$ for $0 \leq j \leq J_n$, meaning that $i\mu_i^n$, $0 \leq j \leq J_n$, are roots of (2.3). One can also easily see that for any fixed $n \leq N$, $(D_m)_M < (D_m)^{J_n}$, if $j, k \leq J_n$ with $j > k$; and for any fixed $j \leq J_n$, $(D_m)_M < (D_m)^{J_n}$, if $m < n \leq N$.

Summarizing the above analysis and noting that $K(1) \geq 2/\pi$ is equivalent to $\beta \leq \exp(\frac{2}{\pi} e^a + 1)$, we obtain the following lemma.

**Lemma 2.8.** The following statements hold.

(i) When $\beta < \exp(\frac{2}{\pi} e^a + 1)$, (2.3) has no purely imaginary root for any $D_m > 0$.

(ii) When $\beta > \exp(\frac{2}{\pi} e^a + 1)$, there exists an $N > 1$ such that for every $n \leq N$, $I_n$ in (2.13) is well-defined. Moreover,

(ii)-1 if $(1 - \ln \beta \cos(\mu_i^n) e^{-at \tau} \leq 1$, then (2.3) has no purely imaginary root for any $D_m > 0$;

(ii)-2 if $(1 - \ln \beta \cos(\mu_i^n) e^{-at \tau} > 1$, then there exists an $J_n \in \mathbb{N}$ with $J_n < I_n$ such that

$$(D_m)^{J_n}$$

is positive for $0 \leq j \leq J_n$, and $i\mu_i^n$, $0 \leq j \leq J_n$, are roots of (2.3) for $D_m = (D_m)^{J_n}$.

In the case of (ii)-2, for any fixed $1 \leq n \leq N$, let $\lambda(D_m) = \xi(D_m) + i\mu(D_m)$ be the root of $\Delta(\lambda, n) = 0$ satisfying $\xi((D_m)^{J_n}) = 0$ and $\mu((D_m)^{J_n}) = \mu_i^n$, when $D_m$ is close to $(D_m)^{J_n}$, for $0 \leq k \leq J_n$. A calculation similar to the proof of lemma 2.2 verifies the transversality condition as stated below.

**Lemma 2.9.** For any fixed $1 \leq n \leq N$, $\xi((D_m)^{J_n}) < 0$, $\forall k = 0, \ldots, J_n$.

By the above two lemmas, we have the following theorem confirming the bifurcation of spatially inhomogeneous periodic solutions, in terms of the diffusion rate $D_m$.

**Theorem 2.10.** In addition to the assumptions for (ii)-2 in lemma 2.8, further assume that except for the pair $\lambda = \pm i\mu_i^n$, there is no other purely imaginary root for the characteristic equation $\Delta(\lambda, n) = 0$ when $D_m = (D_m)^{J_n}$. Then (1.6) undergoes a Hopf bifurcation at $w = \ln \beta$ at $D_m = (D_m)^{J_n}$, and the bifurcating periodic solutions can be written as

$$w(t, x) = \ln \beta + 2\sqrt{\frac{(D_m - (D_m)^{J_n}) \xi(0)}{\text{Re}[\tilde{C}^{nk}_1(0)]}} \rho_n \cos \left(\frac{2\pi t}{\tilde{T}}\right) \cos(n x) + O(D_m - (D_m)^{J_n})$$

with

$$\tilde{T} = \frac{2\pi}{\mu_n} \left[ 1 - \frac{\xi((D_m)^{J_n})(D_m - (D_m)^{J_n})}{\text{Re}[\tilde{C}^{nk}_1(0)]} + O((D_m - (D_m)^{J_n})^2) \right],$$

$$\tilde{\xi}^{nk}_1 = -\frac{1}{\mu_n^2} \left[ \frac{\text{Re}[\tilde{C}^{nk}_1(0)]}{\xi((D_m)^{J_n})} - \frac{\text{Re}[\tilde{C}^{nk}_1(0)]}{\text{Re}[\tilde{C}^{nk}_1(0)]}\mu((D_m)^{J_n}) \right],$$

where $\tilde{C}^{nk}_1(0)$ is given by the same formula (4.22) as for $C^{nk}_1(0)$ in the appendix with $\tilde{p}_n^k$ being replaced by $\beta$ and $D_m$ replaced by $(D_m)^{J_n}$.

**Remark 2.11.** From lemma 2.8-(i), for any fixed $\beta$ and $\tau$, Hopf bifurcation will not occur for large $\alpha = \int_0^T D(a) \, da$. This suggests that large diffusion rate of immature individuals tends to destroy spatially inhomogeneous patterns. Also note that $(D_m)^{J_n} < (D_m)^{J_n}$, for $m < n \leq N$. This implies that the smaller $D_m$ may lead to more complicated spatial patterns.
Table 1. Critical values.

<table>
<thead>
<tr>
<th>n</th>
<th>$\omega_n^0$</th>
<th>$\beta_n^0$</th>
<th>$\beta_n^1$</th>
<th>$\text{Re}C_n^0(0)$</th>
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</thead>
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<td>-</td>
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<td>4</td>
<td>2.4768</td>
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<td>-</td>
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</tr>
</tbody>
</table>

Figure 2. Numerical simulations of (1.6) with $\tau = 3$, $D_m = 0.01$, $\alpha = 0$ and initial function $\ln \beta + 0.2$. (a) $\beta = 1.5$, the solution approaches to the steady state. (b) $\beta = 9.7$, the solution still approaches to the steady state but with noticeable oscillations. (c) $\beta = 9.9$, the solution tends to a spatially homogeneous periodic solution.

3. Numerical analysis and discuss

In this section we present some numerical simulations for (1.6) to illustrate the obtained analytic results.

Firstly, we choose $\alpha = 0$, $\tau = 3$ and $D_m = 0.01$. By using formulas (2.5), (2.6), (4.21) and (4.22), we can obtain table 1 for the related critical values of bifurcations. Note that $\beta_0^1 = 49.9956$, which is greater than all $\beta_n^0$ for $n \leq 4$. It follows that assumption ($H_n^0$) holds for $n \leq 4$. Therefore, from the theoretic results in the last section, we see that for (1.6) the spatially homogeneous steady state is asymptotically stable if $\beta < 9.8976$ and unstable if $\beta > 9.8976$. Moreover, stable spatially homogeneous periodic solutions occur when $\beta$ crosses through 9.8976 and $\cos(nx)$-like, $n = 1, 2, 3, 4$ periodic solutions appear when $\beta$ crosses through 10.0561, 10.5590, 11.4993 and 13.0730, respectively, and they are stable on the centre manifolds.
When $1 < \beta < \beta_0^0$ the spatially homogeneous steady state is asymptotically stable, as shown in figures 2(a) and (b). When $\beta$ crosses through $\beta_0^0$, Hopf bifurcation occurs and the bifurcated periodic solutions are stable as shown in figure 2(c). Besides, the bifurcated periodic solution is spatially homogeneous.

In figure 3, we use the initial function which is a $\cos(x)$-perturbation of the spatially homogeneous steady state. Figure 3(a) plots the solutions at $x = 0$ (blue), $x = \frac{\pi}{2}$ (green) and $x = \frac{2\pi}{3}$ (red) in the time interval $[0, 50]$. Figures 3(b) and (c) are for time intervals $[50, 100]$ and $[750, 800]$, respectively. Figure 3 shows that the solution is spatially inhomogeneous at first, and then it tends to a $\cos(x)$-like periodic solution as shown in figure 3(d). Further, the period of the periodic solution is about 2.56 which is close to the period of the bifurcated periodic solution $\frac{2\pi}{\omega_1} = 2.5572$ (see figure 3(b)). Moreover, the solution approaches to a spatially homogeneous periodic solution with the same period after around time 700.

Since the $\cos(nx)$-like bifurcated periodic solutions are stable on the centre manifolds, we can see from figures 4 and 5(b) that they can be observed for a relatively long period if we choose a perturbation of $\cos(nx)$ as the initial function, when the parameter $\beta$ crosses...
Figure 4. Numerical simulations of (1.6) with $\tau = 3$, $D_m = 0.01$ and $\alpha = 0$. (a) $\beta = 10.6$, initial function $\ln \beta + 0.6 \cos(2x)$, the solution follows a $\cos(2x)$-like pattern for some time and then approaches a uniform periodic solution. (b) $\beta = 11.5$, initial function $\ln \beta + 0.6 \cos(3x)$, the solution follows a $\cos(3x)$-like pattern for some time and then tends to a uniform periodic solution.

Figure 5. Numerical simulations of (1.6) with $\tau = 3$, $D_m = 0.01$ and $\alpha = 0$. (a) $\beta = 11.5$, initial function $\ln \beta + \cos(4x)$, the solution directly approaches a uniform periodic solution. (b) $\beta = 13.5$, the solution follows a $\cos(4x)$-like periodic pattern for some time before approaching a homogeneous periodic solution.

Through the critical values $\beta^0_n$. Figure 5(a) shows that a solution with initial function being a $\cos(4x)$-perturbation of the steady state will not converge to any $\cos(4x)$-like periodic solution, instead it will tend to a spatially homogeneous periodic solution directly when $\beta$ is less than the critical value $\beta^0_n$. Figures 3, 4 and 5 suggest that the spatially homogeneous periodic solution exists for a wide range of the parameter $\beta$ and it is stable.

Now, we explore the effect of the non-locality caused by the mobility of the immature population. To this end, we choose $\alpha = 0.1$, $\tau = 3$ and $D_m = 0.01$. Similarly, we can obtain table 2 for related critical values of bifurcations. From table 2 and the theoretic results we know that: for (1.6) the spatially homogeneous steady state is asymptotically stable if $\beta < 9.8976$ and unstable if $\beta > 9.8976$; $\cos(nx)$-like, $n = 1, 2$, periodic solutions appear when $\beta$ crosses through 11.3744 and 19.0092, respectively, and they are stable in on the corresponding centre manifolds.

In figure 6, we use a $\cos(x)$-perturbation of the spatially homogeneous steady state as the initial function. Figures 6(a) and (b) illustrate that the spatially homogeneous steady state is
Table 2. Critical values.

<table>
<thead>
<tr>
<th>n</th>
<th>$\omega_0^0$</th>
<th>$\beta_0^0$</th>
<th>$\beta_1^1$</th>
<th>Re$C_\infty^0(0)$</th>
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<tbody>
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<td>2.4556</td>
<td>9.8976</td>
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</table>

Figure 6. Numerical simulations of (1.6) with $\tau = 3$, $D_m = 0.01$, $\alpha = 0.1$ and initial function $\ln \beta + 1.5 \cos(x)$. 
(a) $\beta = 9.5$, the solution approaches to the steady state. 
(b) $\beta = 10$, the solution tends to a spatially homogeneous periodic solution. 
(c) $\beta = 15$, the solution first follows a $\cos(x)$-like inhomogeneous periodic solution for some time and then approaches a uniform periodic solution.

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Appendix: Normal form

Assume \((H_1^2)\) holds and let \(\beta = \beta_n^k + \mu, \ \mu \in \mathbb{R}\). Then \(\mu = 0\) is a Hopf bifurcation value for (1.6). For \(\phi \in C\), denote

\[
L_\mu \phi := -\tau \phi(0) + \frac{(1 - \ln(\beta_n^k + \mu))\tau}{\pi} \int_0^\pi \phi(-1) \left[1 + \sum_{n=1}^{+\infty} \left(\cos n(x + y) + \cos n(x - y)e^{-\alpha n^2}\right)\right] dy,
\]

\[
f_\mu \phi := \frac{\tau}{\pi} \int_0^\pi \left[\left(\frac{1}{2} \ln(\beta_n^k + \mu) - 1\right) \phi^2(-1) + \left(\frac{1}{2} - \frac{1}{6} \ln(\beta_n^k + \mu)\right) \phi^3(-1) + O(\phi^4(-1))\right]
\]

\[
\times \left[1 + \sum_{n=1}^{+\infty} \left(\cos n(x + y) + \cos n(x - y)e^{-\alpha n^2}\right)\right] dy,
\]

\[
A(\mu)\phi = \phi + X_0[L_\mu(\phi) + D_{\alpha \phi}(0) - \phi(0)] \text{ and } R(\mu)\phi := X_0f_\mu(\phi), \text{ where } \]

\[
X_0 = \begin{cases} 
0, & \theta \in [-1, 0) \\
1, & \theta = 0.
\end{cases}
\]

Then (1.6) can be rewritten as the following

\[
w(t) = A(\mu)w_1 + R(\mu)w_1,
\]

where “” is the derivative with respect to \(t\) and \(w_1 = w(t + \theta)\) for \(\theta \in [-1, 0]\).

For \(\psi \in C([0, 1], X)\) and \(\phi \in C([-1, 0], X)\), we define a bilinear form as following

\[
\langle \psi, \phi \rangle = \int_0^\pi \psi(0)\phi(0) dy + \int_0^\pi \int_0^\pi \frac{1 - \ln(\beta_n^k)}{\pi} \left[\int_0^\pi \psi(s + 1, z) \right.
\]

\[
\times \left[1 + \sum_{n=1}^{+\infty} \left(\cos m(y + z) + \cos m(y - z)e^{-\alpha n^2}\right)\right] dz \phi(s, y) dy ds.
\]

In the rest of this section, we always use similar notations to those used in Hassard et al [6]. Define \(q_{nk}(\theta) = e^{\frac{\alpha n^2}{2}} \rho_n \cos(nx)\), and \(q_{nk}^*(s) = \bar{P}_n e^{\frac{i\alpha n^2}{2}} \rho_n \cos(nx), n \in \mathbb{N}_0, k \in \mathbb{N}_0\), where

\[
P_n^k := (1 + \tau + D_m n^2 - i\omega_n^k)^{-1}.
\]

Then by direct computations we have

\[
(q_{nk}^*, q_{nk}) = (\bar{q}_{nk}^*, q_{nk}) = 1 \quad \text{and} \quad (q_{nk}^*, \bar{q}_{nk}) = (\bar{q}_{nk}^*, q_{nk}) = 0.
\]

Let \(w_1\) be the solution of (1.6) when \(\mu = 0\) and define \(z_{nk}(t) = (q_{nk}, w_1)\) and \(W_{nk}(t, \theta, x) = w_1(t - \theta) - 2\Re(z_{nk}(t)q_{nk})\). Using the definitions of \(z_{nk}\), \(W_{nk}\) and the bilinear form, it is not difficult to verify that (1.6) is reduced to the following system:

\[
\begin{cases}
(z_{nk}'(t)) = i\omega_n^k z_{nk} + \int_0^\pi \bar{q}_{nk}(0) f_0 dy,

(W_{nk}'(t)) = A(0) W_{nk} - 2\Re\left\{\int_0^\pi \bar{q}_{nk}(0) f_0 dy q_{nk}(\theta)\right\} + X_0 f_0, \ \theta \in [-1, 0],
\end{cases}
\]

where \(f_0 = f_0(2\Re(z_{nk}(t)q_{nk}) + W(t, \theta))\). Denote

\[
g_{nk}(z_{nk}, \bar{z}_{nk}) = \int_0^\pi \bar{q}_{nk}(0) f_0 dy := g_{20}^{nk} z_{nk}^2 + g_{11}^{nk} z_{nk} \bar{z}_{nk} + g_{02}^{nk} \bar{z}_{nk}^2 + g_{21}^{nk} z_{nk} \bar{z}_{nk} + \cdots.
\]
Then, the Poincaré normal form for (1.6) has the following form:

\[ z'_{nk} = \lambda(\mu)z + C^{nl}_{1k}(\mu)z^2\bar{z} + \text{h.o.t.,} \quad (4.5) \]

and

\[ C^{nl}_{1k}(0) = \frac{i}{2\omega^b_{nk}} \left( g^{nl}_{0k}g_{11}^{nl} - 2|g^{nl}_{11}|^2 - \frac{1}{3}|g^{nl}_{02}|^2 \right) + \frac{g^{nl}_{21}}{2}. \quad (4.6) \]

To obtain the existence of the non-trivial periodic solutions, the only remaining thing is the calculations of \( C^{nl}_{1k} \).

Using the centre manifold theorem given in [3], we know that on the centre manifold \( W_{nk}(t, \theta) \) has the following form

\[ W_{nk} = W_{nk}^{20}(\theta, x)z_{nk}^2 + W_{nk}^{11}(\theta, x)z_{nk}\bar{z}_{nk} + W_{nk}^{02}(\theta, x)\bar{z}_{nk}^2 + \cdots. \quad (4.7) \]

By expanding the series and comparing the corresponding coefficients, we have

\[ g^{nl}_{20} = \begin{cases} \frac{2\tau}{\sqrt{\pi}} \left( 6 \right) \left( \frac{1}{2} \ln \beta_0^k - 1 \right) \cos\left( \frac{n}{2} \omega_k^b \right) e^{-2i\omega_k^b}, & n = 0 \\ 0, & n \neq 0, \end{cases} \quad (4.8) \]

\[ g^{nl}_{02} = \begin{cases} \frac{2\tau}{\sqrt{\pi}} \left( 6 \right) \left( \frac{1}{2} \ln \beta_0^k - 1 \right) \sin\left( \frac{n}{2} \omega_k^b \right), & n = 0 \\ 0, & n \neq 0, \end{cases} \quad (4.9) \]

\[ g^{nl}_{11} = \begin{cases} \frac{2\tau}{\sqrt{\pi}} \left( 6 \right) \left( \frac{1}{2} \ln \beta_0^k - 1 \right) \cos\left( \frac{n}{2} \omega_k^b \right), & n = 0 \\ 0, & n \neq 0. \end{cases} \quad (4.10) \]

\[ g^{nl}_{21} = \frac{2\tau}{\pi} \left( 6 \right) \left( \frac{1}{2} \ln \beta_0^k - 1 \right) \int_0^\pi \cos\left( \frac{n}{2} \omega_k^b \right) W_{nk}^{11}(\theta, x) d\theta + \frac{1}{2} \sum_{|m| = 1}^\infty \left( \cos m(y + z) + \cos m(y - z) \right) e^{-\alpha m^2} d\theta d\phi - \frac{i}{2} \rho_{nk}^k \left( \frac{1}{2} - \frac{1}{6} \ln \beta_0^k \right) e^{-\alpha n^2}, \quad n \neq 0. \quad (4.11) \]

and

\[ g^{nl}_{21} = \frac{8\tau}{\pi^2} \left( 6 \right) \left( \frac{1}{2} \ln \beta_0^k - 1 \right) \int_0^\pi \cos\left( \frac{n}{2} \omega_k^b \right) \int_0^\pi \cos\left( \frac{n}{2} \omega_k^b \right) W_{nk}^{11}(\theta, x) d\theta + \frac{1}{2} \sum_{|m| = 1}^\infty \left( \cos m(y + z) + \cos m(y - z) \right) e^{-\alpha m^2} d\theta d\phi - \frac{i}{2} \rho_{nk}^k \left( \frac{1}{2} - \frac{1}{6} \ln \beta_0^k \right) e^{-\alpha n^2}, \quad n \neq 0. \quad (4.12) \]

Next we calculate the centre manifold. Let

\[ H_{nk}(\bar{z}_{nk}, \bar{z}_{nk}, \theta) = -2\text{Re} \left\{ \int_0^\pi \tilde{g}_{nk}^k(0) f_0 d\phi q_{nk}(\theta) \right\} + X_0 f_0 \quad (4.13) \]

where

\[ H_{nk}(\bar{z}_{nk}, \bar{z}_{nk}, \theta) = H_{20}^{nk}(\theta) \bar{z}_{nk}^2 + H_{11}^{nk}(\theta) \bar{z}_{nk}^2 + H_{02}^{nk}(\theta) \bar{z}_{nk}^2 + H_{21}^{nk}(\theta) \bar{z}_{nk}^2 + \cdots. \]

Expanding both sides of the second equation of (4.3) and comparing the corresponding coefficients, we can obtain that

\[ (2i\omega^b_{nk} - A(0)) W_{nk}^{20} = H_{20}^{nk}, \quad (4.14) \]

\[ A(0) W_{nk}^{11} = -H_{11}^{nk}, \quad (4.15) \]

\[ (2i\omega^b_{nk} + A(0)) W_{nk}^{02} = -H_{02}^{nk}. \quad (4.16) \]
By comparing coefficients of the both sides of (4.13), we have that for $\theta \in [-1, 0)$,

\[
\begin{align*}
H_{20}^{10}(\theta) &= -s_{20}^{10} g_{nk}(\theta) - \frac{s_{02}^{10}}{s_{02}^{10}} q_{nk}(\theta), \\
H_{11}^{10}(\theta) &= -s_{11}^{10} g_{nk}(\theta) - \frac{s_{11}^{10}}{s_{11}^{10}} q_{nk}(\theta).
\end{align*}
\]

Substituting the above equations into (4.14) and (4.15) respectively, leads to

\[
\begin{align*}
W_{20}^{nk}(\theta) &= 2 i \omega_k \omega_n W_{20}^{nk}(\theta) + g_{20}^{nk} \rho_n \cos (nx) e^{i\omega_n \theta} + \frac{\pi}{s_{02}^{10}} \rho_n \cos (nx) e^{-i\omega_n \theta}, \\
W_{11}^{nk}(\theta) &= g_{11}^{nk} \rho_n \cos (nx) e^{i\omega_n \theta} + \frac{s_{11}^{10}}{s_{11}^{10}} \rho_n \cos (nx) e^{-i\omega_n \theta}.
\end{align*}
\]

Solving the above equations, we obtain,

\[
\begin{align*}
W_{20}^{nk}(\theta) &= -\frac{s_{20}^{10}}{s_{20}^{10} \rho_n} \cos (nx) e^{i\omega_n \theta} - \frac{s_{02}^{10}}{3i \omega_n} \rho_n \cos (nx) e^{-i\omega_n \theta} + E^{nk}(x) e^{2i\omega_n \theta}, \\
W_{11}^{nk}(\theta) &= \frac{s_{11}^{10}}{s_{11}^{10} \rho_n} \rho_n \cos (nx) e^{-i\omega_n \theta} - \frac{s_{11}^{10}}{s_{11}^{10} \rho_n} \rho_n \cos (nx) e^{-i\omega_n \theta} + F^{nk}(x),
\end{align*}
\]

where $E^{nk}(x)$ and $F^{nk}(x)$ can be expressed as $E^{nk}(x) = \sum_{m=0}^{\infty} E_{m}^{nk} \cos (nx)$ and $F^{nk}(x) = \sum_{m=0}^{\infty} F_{m}^{nk} \cos (nx)$, respectively, and $E_{m}^{nk}$ and $F_{m}^{nk}$ $(m, n \in \mathbb{N}_0)$ are constants for fixed parameters. However,

\[
\begin{align*}
H_{20}^{0k} &= - \frac{s_{20}^{10}}{s_{20}^{10} \rho_n} \cos (nx) + \frac{s_{02}^{10}}{s_{02}^{10} \rho_n} \cos (nx) + \frac{2 \pi}{\rho_n} \left( \frac{1}{2} \ln \beta - 1 \right) e^{-2i\omega_n \theta}, \\
H_{11}^{0k} &= - \frac{s_{11}^{0k}}{s_{11}^{0k} \rho_n} \cos (nx) - \frac{s_{11}^{0k}}{s_{11}^{0k} \rho_n} \cos (nx) + \frac{2 \pi}{\rho_n} \left( \frac{1}{2} \ln \beta - 1 \right) e^{-2i\omega_n \theta}, \\
H_{20}^{1k} &= \frac{s_{20}^{10}}{s_{20}^{10} \rho_n} \cos (nx) + \frac{s_{20}^{10}}{s_{20}^{10} \rho_n} \cos (nx) + \frac{2 \pi}{\rho_n} \left( \frac{1}{2} \ln \beta - 1 \right) e^{-2i\omega_n \theta}, \\
H_{11}^{1k} &= \frac{s_{11}^{0k}}{s_{11}^{0k} \rho_n} \cos (nx) - \frac{s_{11}^{0k}}{s_{11}^{0k} \rho_n} \cos (nx) + \frac{2 \pi}{\rho_n} \left( \frac{1}{2} \ln \beta - 1 \right) e^{-2i\omega_n \theta}.
\end{align*}
\]

Substituting (4.17) and (4.19) into (4.14) and (4.15), respectively, we then have for $n = 0$,

\[
E^{0k} = \frac{2 \pi}{\rho_n} \left( \frac{1}{2} \ln \beta - 1 \right) e^{-2i\omega_n \theta} + \sum_{p=1}^{\infty} E_{p}^{0k} \cos (px)
\]

and

\[
P^{0k} = \frac{2 \pi}{\rho_n} \left( \frac{1}{2} \ln \beta - 1 \right) + \sum_{p=1}^{\infty} P_{p}^{0k} \cos (px).
\]

Similarly, we can obtain that for $n > 0$,

\[
E^{nk} = \frac{2 \pi}{\rho_n} \left( \frac{1}{2} \ln \beta - 1 \right) e^{-2i\omega_n \theta} + \sum_{p=1}^{\infty} E_{p}^{nk} \cos (px)
\]

and

\[
P^{nk} = \frac{2 \pi}{\rho_n} \left( \frac{1}{2} \ln \beta - 1 \right) + \sum_{p=1}^{\infty} P_{p}^{nk} \cos (px).
\]
and 
\[ F_{nk} = \frac{2(\frac{1}{2} \ln \beta_n - 1)}{\pi \ln \beta_n} \left( 1 - (1 - \ln \beta_n) e^{-4an^2}\cos(2nx) \right) + \frac{2\tau (\frac{1}{2} \ln \beta_n - 1) e^{-4an^2} \cos(2nx)}{\pi[4D_m n^2 + \tau - (1 - \ln \beta_n) \tau e^{-4an^2}]} + \sum_{p=1, p \neq 2n}^{+\infty} F_p \cos(px). \]

Then, we have
\[ g_{21}^{nk} = \frac{\tau}{\pi} \bar{P}_n \left( 3 - \ln \beta_n \right) e^{-2\tau \omega_k} + 4\tau^2 \bar{P}_n \left( \frac{1}{2} \ln \beta_n - 1 \right)^2 \left( \frac{\beta_n}{\omega_n} e^{-2\tau \omega_k} - \frac{7P_n}{3i\omega_n} \right), \]
\[ g_{21}^{nk} = 2\tau^2 \bar{P}_n \left( \frac{1}{2} \ln \beta_n - 1 \right)^2 e^{-\omega_k} \left[ \frac{2e^{-3i\omega_k}}{\pi[2i\omega_n + \tau - (1 - \ln \beta_n) \tau e^{-2i\omega_k}]} + \frac{2e^{-3i\omega_k}}{\omega_k \ln \beta_n} \right] \]
\[ + \frac{\tau^2 \bar{P}_n \left( \frac{1}{2} \ln \beta_n - 1 \right)^2 e^{-\omega_k}}{\omega_k \ln \beta_n} \left[ \frac{2e^{-3i\omega_k}}{\pi[2i\omega_n + 4D_m n^2 + \tau - (1 - \ln \beta_n) \tau e^{-2i\omega_k} e^{-4an^2}]} + \frac{4e^{-\omega_k}}{\pi[4D_m n^2 + \tau - (1 - \ln \beta_n) \tau e^{-2i\omega_k} e^{-4an^2}]} \right] \]
\[ \text{for } n \neq 0. \]

Therefore,
\[ C_i^{nk}(0) = 2\tau^2 \bar{P}_n \left( \frac{1}{2} \ln \beta_n - 1 \right)^2 \left[ \frac{\omega_k}{\pi[2i\omega_n + \tau - (1 - \ln \beta_n) \tau e^{-2i\omega_k}]} + \frac{2e^{-3i\omega_k}}{\pi \ln \beta_n} \right] \]
\[ + \frac{\tau}{2\pi} \bar{P}_n \left( 3 - \ln \beta_n \right) e^{-2\tau \omega_k}, \]
\[ \text{for } n \neq 0. \]

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