Threshold Dynamics in a Delayed SIS Epidemic Model

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An SIS epidemic model with maturation delay is analysed. It is shown that the disease dies out when the basic reproduction number $R_0 < 1$, and the disease remains endemic when $R_0 > 1$ in the sense of uniform persistence. When the disease induced death rate is sufficiently small, the global attractivity of the endemic equilibrium is also proved.

Key Words: maturation delay; epidemic model; threshold dynamics; persistence; global attractivity.

1. INTRODUCTION

In their recent paper, Cooke et al. [1] derived a population growth model for single-species with multiple life stages and came up with a delay differential equation

$$ N'(t) = B(N(t - T))N(t - T)e^{-dT} - dN(t), $$

(1.1)

where $' = \frac{d}{dt}$, $N(t)$ is the adult (matured) population size at time $t$, $d > 0$ is the death rate constant at the adult stage, $B(N)$ is a birth rate function, $T$ is the developmental or maturation time, and $d_1$ is the death rate constant for each life stage prior to the adult stage. Typical examples of birth rate functions $B(N)$ found in the biological literature are:

(B1) $B_1(N) = be^{-aN}$, with $a > 0$, $b > d$;
(B2) $B_2(N) = p/(q + N^n)$, with $p, q, n \geq 0$, and $\frac{p}{q} > d$;
(B3) $B_3(N) = A/N + c$, with $A > 0$, $d > c > 0$.

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Functions $B_1$ and $B_2$ with $n = 1$ are used in fisheries and are known as the Ricker function and the Beverton–Holt function, respectively. Function $B_3(N)N$ represents a constant immigration rate $A$ together with a linear birth term $cn$.

There have been many models for single-species population growth, which have taken into consideration the maturation delay, and most of these models basically fall into the following two types:

$$N'(t) = N(t)f(N(t), N(t - T)),$$

$$N'(t) = B(N(t - T))N(t - T) - dN(t) \quad (1.2)$$

$$N'(t) = \frac{I}{N} - (d + \epsilon + \gamma)I \quad \epsilon \geq 0 \text{ is the disease induced death rate constant, } \gamma \geq 0 \text{ is the recovery rate constant (} \frac{1}{\tau} \text{ is the average infective time), and } \lambda > 0 \text{ is the contact rate constant. The standard incidence function is used with } \frac{I}{N} \text{ giving the average number of adequate contacts with infectives of one susceptible per unit time. For some diseases, this incidence function seems to fit the data better than mass action incidence; see Mena-Lorca and Hethcote [8], de Jong et al. [2], and the references therein. This model is obtained under the following assumptions:}

(A1) transmission of the disease occurs due to contact between susceptibles and infectives;

(A2) there is no vertical transmission;

(A3) the disease confers no immunity, and thus upon recovery an infective individual returns to the susceptible class (hence the name SIS model).
This type of model is appropriate for some bacterial infections. For a fatal
disease, the recovery rate constant is set to zero, giving an SI model.

Because of the relation \( N = I + S \), for (1.4) it is sufficient to consider

\[
I'(t) = \lambda (N-I) \frac{I}{N} - (d + \varepsilon + \gamma) I
\]

\[
N'(t) = B(N(t-T)) N(t-T) e^{-d;T} - dN - \varepsilon I.
\]

The global dynamics of Eq. (1.1) and system (1.5) were studied in Cooke
et al. [1]. For (1.5), a basic reproduction number was identified, that is,

\[
R_0 = \frac{\lambda}{d + \varepsilon + \gamma},
\]

which gives the average number of new infectives produced by one
infective during the mean death adjusted infective period. When there is
no delay \((T = 0)\), \(R_0\) acts as a sharp threshold, as shown in Cooke et al. [1,
Theorem 4.1]. When \(T > 0\), \(R_0 < 1\) implies the existence of an unique
non-trivial equilibrium called disease free equilibrium, which is globally
asymptotically stable (Cooke et al. [1, Theorems 4.2–4.3]). In the case of
\(R_0 > 1\), there exists also an endemic equilibrium, in addition to the disease
free one, and analysing the dynamics of (1.5) in this case becomes quite
hard in general. For those \(B(N)\) with \(\frac{d}{\pi e}(B(N)N) > 0\), Cooke et al. [1,
Theorems 4.4–4.5] obtained the globally asymptotic stability of the en-
demic equilibrium for \(\varepsilon > 0\) and established the locally asymptotic stability
for \(\varepsilon > 0\) but with either \(B(N) = B_1(N)\) or \(B(N) = B_2(N)\). As for the
general \(B(N)\) and \(\varepsilon > 0\), the dynamics of (1.5) remains undetermined, and
this paper makes an attempt in this direction. In other words, we will
investigate the threshold dynamics of system (1.5) for more general \(B(N)\)
satisfying some biologically reasonable conditions, and we will allow \(\varepsilon > 0\).
Our approach will be a combination of the theory of monotone dynamical
systems, theory of asymptotically autonomous semiflows, some abstract
persistence theorems, and a perturbation technique.

2. THRESHOLD DYNAMICS

We first impose the following conditions on (1.5):

\((H1)\) \(B(\cdot) \in C'([0, \infty), (0, \infty))\) with \(B'(N) < 0, \forall N \in (0, \infty), B(0^+) > (d + \varepsilon)e^{d;T}\) and \(de^{d;T} > B(\infty)\); and there exists a \(G(.) \in C^1([0, \infty), R)\) such that
\(G(N) = B(N)N, \forall N > 0\).

\((H2)\) Either \(B(N)N > 0, \forall N \in (0, \infty)\), or \(B(N)N\) is bounded on
\((0, \infty)\) and the positive equilibrium \(N_e = B^{-1}(de^{d;T})\) of (1.1) is globally
asymptotically stable for initial values in \(C([-T, 0], R^+ \setminus \{0\})\).
Note that the condition \((B(N)N)\mathcal{Y} > 0, \forall N \in (0, \infty)\), is sufficient for the global asymptotic stability of \(N^*_e\) for (1.1) with initial values in \(C([-T, 0], R^+)\) (see Cooke et al. [1, Theorem 3.1]). For other sufficient conditions, we refer to Cooke et al. [1, Theorem 3.3 (iii) and (iv)].

In order to get the existence, uniqueness, and positive invariance of solutions of (1.5), we define a function \(F \in C(R^+_2, R)\) by

\[
F(0, 0) = 0, \quad F(x, y) = \frac{xy}{x + y} \quad \text{if} \ (x, y) \in R^+_2 \setminus \{(0, 0)\}.
\]

It is easy to verify that \(F\) is globally Lipschitz on \(R^+_2\),

\[
|F(x_1, y_1) - F(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2|,
\]

\(\forall (x_i, y_i) \in R^+_2, i = 1, 2\).

Using \(S = N - I\), system (1.5) is transformed into the following nonlinear delayed system

\[
\begin{align*}
I'(t) &= -(d + \epsilon + \gamma)I + \lambda F(I, S) \\
S'(t) &= \gamma I - dS - \lambda F(I, S) \\
&\quad + G(I(t - T) + S(t - T))e^{-dT}.
\end{align*}
\]

By Smith [10, Theorem 5.2.1], for any \(\phi \in C_+ = C([-T, 0], R^+_2)\), there is a unique solution \((I(t, \phi), S(t, \phi))\) of (2.1) with \((I(\theta, \phi), S(\theta, \phi)) = \phi(\theta)\), \(\forall \theta \in [-T, 0]\), and \(I(t, \phi) \geq 0, S(t, \phi) \geq 0\) for all \(t \geq 0\) in its maximal interval of existence. Then \(N(t) = S(t) + I(t)\) satisfies the differential inequality

\[
N'(t) \leq G(N(t - T))e^{-dT} - dN.
\]

By Smith [10, Theorem 5.1.1] and the standard comparison theorem, together with assumption (H2), it then follows that each solution \((I(t, \phi), S(t, \phi))\) exists globally on \([0, \infty)\) and solutions of (2.1) are ultimately bounded. Moreover, if \(\phi = (\phi_1, \phi_2) \in C_+ \) with \(\phi_i(0) > 0\), by using the two equations in (2.1) respectively, we then have \(I(t, \phi) > 0, S(t, \phi) > 0, \forall t \geq 0\).

Let \(X = \{(\phi_1, \phi_2) \in C_+ : \phi_2(\theta) \geq \phi_1(\theta), \forall \theta = [-T, 0]\}, X_0 = \{(\phi_1, \phi_2) \in X : \phi_1(0) > 0\}\) and \(\partial X_0 = X \setminus X_0\). Clearly, \(X_0\) is an open set relative to \(X\). To consider the dynamics of (1.5), we extend it to the system

\[
\begin{align*}
I'(t) &= \lambda F(I, N - I) - (d + \epsilon + \gamma)I, \\
N'(t) &= G(N(t - T))^{-d} - dN - \epsilon I.
\end{align*}
\]
By the aforementioned conclusions for (2.1), it then follows that for any \( \phi \in X \), the unique solution \((I(t, \phi), N(t, \phi))\) of (2.2) with \((I(\theta, \phi), N(\theta, \phi)) = \phi(\theta), \forall \theta \in [-T, 0]\) satisfies \(I(t, \phi), N(t, \phi) \geq 0, \forall t \in [0, \infty)\). Let \(\Phi(t) : X \rightarrow X, t \geq 0\), be the solution semiflow generated by (2.2), that is, \((\Phi(t)\phi)(\theta) = (I(t + \theta, \phi), N(t + \theta, \phi)), \theta \in [-T, 0], t \geq 0\). Thus we further have \(\Phi(t) : X_0 \rightarrow X_0, \Phi(t) : \partial X_0 \rightarrow \partial X_0, \forall t \geq 0\), and \(\Phi(t) : X \rightarrow X\) is point dissipative. By the standard theory of FDE (see, e.g., Hale and Verduyn Lunel [6]), \(\Phi(t) : X \rightarrow X\) is compact for each \(t > T\), and hence, there is a global attractor \(A\) for \(\Phi(t) : X \rightarrow X\) (see Hale [5, Theorem 3.4.8]).

Now we are in the position to state and prove the following result of threshold dynamics for (1.5), determining whether the disease dies out or remains endemic eventually.

**THEOREM 2.1.** Assume that (H1) and (H2) hold. If \(R_0 < 1\), then every solution \((I(t, \phi), N(t, \phi))\) of (1.5) with \(\phi \in X_0\) satisfies \(\lim_{t \rightarrow \infty} I(t, \phi) = 0\) and \(\lim_{t \rightarrow \infty} N(t, \phi) = N_c\). If \(R_0 > 1\), then there is a \(\beta > 0\) such that every solution \((I(t, \phi), N(t, \phi))\) of (1.5) with \(\phi \in X_0\) satisfies \(\liminf_{t \rightarrow \infty} N(t, \phi) \geq \lim \inf_{t \rightarrow \infty} I(t, \phi) \geq \beta\).

**Proof.** Let \(B(\cdot), \lambda, d, d_1, \gamma, T\) be fixed. Then we have the following claim.

**Claim 1.** For any positive number \(e_0\) with \(B(0^+) > (d + e_0)e^{d_1T}\), there exists \(\delta_1 = \delta_1(e_0) > 0\) such that for any \(e \in [0, e_0]\), the solution semiflow \(\Phi(t)\) associated with (2.2) satisfies \(\limsup_{t \rightarrow \infty} \|\Phi(t)\phi\| \geq \delta_1, \forall \phi \in X_0\).

Indeed, let \(\eta_l = \frac{1}{2}(B(0^+) - (d + e_0)e^{d_1T})\). Note that if \(B(0^+) = \infty\), we replace \(B(0^+)\) with any positive number \(B^+ > (d + e_0)e^{d_1T}\). Then there exists a \(\delta_1 = \delta_1(\eta_l)\) such that \(B(N) > B(0^+) - \eta_l > 0, \forall N < \delta_1\). Suppose that, by contradiction, \(\limsup_{t \rightarrow \infty} \|\Phi(t)\phi\| < \delta_1\) for some \(\phi \in X_0\) and \(e \in [0, e_0]\). Then there exists a \(T_1 > 0\) such that \(\|\Phi(t)\phi\| < \delta_1, \forall t \geq T_1\). Let \((I(t), N(t)) = (\Phi(t)\phi)(0), t \geq 0\). Thus \(N(t) \geq I(t), \forall t \geq 0\), and hence \(N(t)\) satisfies the differential inequality

\[
N'(t) \geq (B(0^+) - \eta_l)N(t - T)e^{-d_1T} - (d + e)N(t), \quad \forall t > T_1.
\]

Consider the linear delayed equation

\[
N'(t) = (B(0^+) - \eta_l)N(t - T)e^{-d_1T} - (d + e)N(t), \quad t \geq T_1.
\]

(2.3)

Since \(B(0^+) - \eta_l > 0\), (2.3) is a cooperative and irreducible equation, and hence, by Smith [10, Corollary 5.5.2], the linear stability of \((0, 0)\) for (2.3) is the same as for ordinary differential equation (by ignoring the delay in
\( N(t - T) \)

\[
N'(t) = (B(0^+) - \eta_1)e^{-d_1T} - (d + \epsilon)N(t), \quad t \geq T_1. \tag{2.4}
\]

Note that \( \eta_1 < B(0^+) - (d + \epsilon_0)e^{-d_1T} \) and \( \epsilon \in [0, \epsilon_0] \). We then have \((B(0^+) - \eta_1)e^{-d_1T} - (d + \epsilon) > 0\), and hence, by Smith [10, Corollary 5.5.2], the stability modulus \( s \) of (2.3) is positive. Then, by Smith [10, Theorem 5.5.1], (2.3) admits a solution \( N^*(t) = e^{st}u \) with \( u > 0 \). Since \( N(t) > 0, \forall t > 0 \), we can choose a \( k > 0 \) such that \( N(t) > kN^*(t), \forall t \in [T_1 - T, T_1] \). By the comparison theorem of quasimonotone systems with delays (Smith [10, Theorem 5.1.1]), we get \( N(t) \geq kN^*(t), \forall t \geq T_1 \), and hence \( \lim_{t \to \infty} N(t) = \infty \), which contradicts the boundedness of \((I(t), N(t))\) on \([0, \infty)\).

In the case of \( R_0 < 1 \), we have \( \lambda < d + \epsilon + \gamma \). If \( I(0) > 0 \), then \( N(t) \geq I(t) > 0, \forall t \geq 0 \), and hence, by Eq. (1.5), we get

\[
I'(t) \leq (\lambda - (d + \epsilon + \gamma))I(t), \quad t \geq 0.
\]

It then follows that

\[
I(t) \leq I(0)e^{(\lambda-(d+\epsilon+\gamma)t)}, \quad t \geq 0,
\]

and hence \( \lim_{t \to \infty} I(t) = 0 \) exponentially. Thus \( N(t) \) satisfies the following non-autonomous delayed equation

\[
N'(t) = B(N(t - T))N(t - T)e^{-d_1T} - dN - \epsilon I(t),
\]

which is asymptotic to the autonomous delayed equation (1.1). By the generalized Markus’ theorem for asymptotically autonomous semiflows (see Thieme [14, Theorem 4.1]), together with assumption (H2) and Claim 1 above, it then follows that \( \lim_{t \to \infty} N(t) = N^*_\epsilon \).

In the case of \( R_0 > 1 \), that is, \( \lambda > d + \epsilon + \gamma \), let \( M_1 = (0, 0) \) and \( M_2 = (0, N_\epsilon) \). By assumption (H2), we have \( \tilde{A}_\phi := \bigcup_{\phi \in \partial X_\phi} \omega(\phi) = \{M_1, M_2\} \), where \( \omega(\phi) \) is the omega limit set of \( \phi \) for the solution semiflow \( \Phi(t) \). Clearly, \( M_1 \) and \( M_2 \) are disjoint, compact, and isolated invariant sets for the semiflow \( \Phi(t)|_{\partial X_\phi} \), and no subset of \( \{M_1, M_2\} \) forms a cycle in \( \partial X_\phi \). We further have the following claim.

Claim 2. For any positive number \( \epsilon_0 \) with \( \lambda > d + \epsilon_0 + \gamma \), there exists \( \delta_2 = \delta_2(\epsilon_0) > 0 \) such that for any \( \epsilon \in [0, \epsilon_0] \), the solution semiflow \( \Phi(t) \) associated with (2.2) satisfies \( \lim_{t \to \infty} \|\Phi(t)\phi - (0, N_\epsilon)\| \geq \delta_2, \forall \phi \in X_\phi \).

Indeed, let \( \eta_2 = \frac{1}{\lambda}(1 + (d + \epsilon_0 + \gamma)/\lambda) \). Since \( \lim_{(t,N) \to (0,N_\epsilon)} \frac{N - \tilde{N}}{N_\epsilon} = 1 > \eta_2 \), there exists a \( \delta_2 = \delta_2(\eta_2) \) such that \( \frac{N - \tilde{N}}{N_\epsilon} > \eta_2, \forall 0 \leq I < \delta_2, |N - \tilde{N}| < \delta_2 \). Suppose that, by contradiction, \( \lim_{t \to \infty} \|\Phi(t)\phi - (0, N_\epsilon)\| < \delta_2 \) for some \( \phi \in X_\phi \) and \( \epsilon \in [0, \epsilon_0] \). Then there exists a \( T_2 > 0 \) such that \( \|\Phi(t)\phi - (0, N_\epsilon)\| < \delta_2, \forall t \geq T_2 \), let \( (I(t), N(t)) = (\Phi(t)\phi)(0), t \geq 0 \). Thus \( I(t) \) satisfies the differential inequality

\[
I'(t) \geq (\lambda \eta_2 - (d + \epsilon + \gamma))I(t), \quad \forall t \geq T_2.
\]
Then

\[ I(t) \geq I(T_2) e^{(\lambda \eta_2 - (d + \epsilon + \gamma) \lambda T_2)}, \quad \forall t \geq T_2. \]

By the choice of \( \eta_2 \), we have \( \lambda \eta_2 - (d + \epsilon + \gamma) \geq \lambda \eta_2 - (d + \epsilon_0 + \gamma) > 0 \), and hence \( \lim_{t \to \infty} I(t) = \infty \), which contradicts the boundedness of \( (I(t), N(t)) \) on \([0, \infty)\).

In view of Claims 1 and 2 above, it is easy to see that \( M_1 \) and \( M_2 \) are isolated invariant sets for \( \Phi(t) \) in \( X \), and \( W'(M_i) \cap X_0 = \emptyset \), \( i = 1, 2 \), where \( W'(M_i) \) is the stable set of \( M_i \) for \( \Phi(t) \). By the acyclicity theorem on uniform persistence (see Hale and Waltman [7, Theorem 4.1]), for any positive \( \epsilon \) satisfying (H1), the semiflow \( \Phi(t) \) is uniformly persistent with respect to \( (X_0, \partial X_0) \) in the sense that there is a \( \beta_0 > 0 \) such that \( \liminf_{t \to \infty} \text{dist} (\Phi(t) \phi, \partial X_0) \geq \beta_0 \) for all \( \phi \in X_0 \). Then, by Smith and Zhao [13, Theorem A.2] with \( Z = C([-T, 0], R^2) \) and \( e = (1, 1) \), there exists a \( \beta > 0 \) such that \( (I(t, \phi), N(t, \phi)) = (\Phi(t) \phi)(0), \; t \geq 0 \), satisfies \( \liminf_{t \to \infty} I(t, \phi) \geq \beta \) and \( \liminf_{t \to \infty} N(t, \phi) \geq \beta \) for all \( \phi \in X_0 \).

In the case that the disease induced death rate \( \epsilon \) is sufficiently small and the basic reproduction number \( R_0 > 1 \), we have the following result on the global attractivity of the endemic equilibrium.

**Theorem 2.2.** Assume that (H1) with \( \epsilon = 0 \) and (H2) hold. If \( \frac{\lambda}{d + \gamma} > 1 \), then there exists an \( \bar{\epsilon} > 0 \) such that for any \( \epsilon \in [0, \bar{\epsilon}] \), system (1.5) admits a positive equilibrium \( (I^\epsilon(\epsilon), N^\epsilon(\epsilon)) \) which is globally attractive in \( X_0 \).

**Proof.** By assumption, we can choose an \( \epsilon_0 > 0 \) sufficiently small such that \( B(0^+) > (d + \epsilon)e^{-dT} \) and \( R_0 = \lambda/(d + \epsilon_0 + \gamma) > 1 \). Note that for any \( \phi \in X \), the solution (\( I(t, \phi), N(t, \phi) \)) of (2.2) satisfies \( N(t) \geq I(t) \geq 0 \), \( t \geq 0 \), and

\[ N'(t) \leq G(N(t - T))e^{-d_1T} - dN(t), \quad t \geq 0. \tag{2.5} \]

Since \( B(\infty) < de^{-d_1T} \), there exists a \( K_0 > 0 \) such that \( B(N)e^{-d_1T} - d < 0 \), \( \forall N \geq K_0 \). Then Smith [10, Theorem 5.2.1 and Remark 5.2.1] imply that any interval \([0, K] \), \( K \geq K_0 \), is positively invariant for the scalar delayed equation

\[ N'(t) = G(N(t - T))e^{-d_1T} - dN(t), \quad t \geq 0. \tag{2.6} \]

In the case that \( B(N)N \) is bounded by \( K_1 \) on \((0, \infty)\), \( N(t, \phi) \) also satisfies

\[ N'(t) \leq K_1e^{-d_1T} - dN(t), \quad t \geq 0. \tag{2.7} \]

By the comparison theorem for quasimonotone systems with delays (Smith [10, Theorem 5.1.1]) and the standard comparison theorem for scalar ordinary differential equations, it then follows that solutions of (2.2) are uniformly bounded and ultimately bounded uniformly for \( \epsilon \in [0, \epsilon_0] \). Let
\( \Phi(\epsilon, t) \) be the solution semiflow generated by (2.2). As mentioned before, for each \( \epsilon \in [0, \epsilon_0] \) and \( t > T \), \( \Phi(\epsilon, t) : X \to X \) is compact. It then easily follows that for any fixed \( t > T \), \( \Phi(\cdot, t) \phi : [0, \epsilon_0] \to X \) is continuous uniformly for \( \phi \) in any bounded subset \( B \) of \( X \), and hence, \( \bigcup_{\epsilon \in [0, \epsilon_0]} \Phi(\epsilon, t)B \) is compact in \( X \) (see, e.g., the Claim in the proof of Smith and Zhao [12, Theorem 3.1]). Moreover, there is a bounded and closed subset \( B_0 \) of \( X \), independent of \( \epsilon \in [0, \epsilon_0] \), such that for any \( \phi \in X \), \( \epsilon \in [0, \epsilon_0] \), there exists \( t_0 = t_0(\phi, \epsilon) > 0 \) such that \( \Phi(\epsilon, t) \phi \in B_0 \) for all \( t \geq t_0 \). Let \( \omega_t(\phi) \) be the omega limit set of \( \phi \in X \) for \( \Phi(\epsilon, t) : X \to X \). Clearly, \( \omega_t(\phi) \) is invariant for \( \Phi(\epsilon, t) \) and is a subset of \( B_0 \). Then,

\[
\bigcup_{\epsilon \in [0, \epsilon_0], \phi \in X} \omega_t(\phi) \subseteq \bigcup_{\epsilon \in [0, \epsilon_0], \phi \in X} \Phi(\epsilon, t)(\omega_t(\phi)) \subseteq \bigcup_{\epsilon \in [0, \epsilon_0]} \Phi(\epsilon, t)B_0, \quad \forall t > T,
\]

and hence \( \bigcup_{\epsilon \in [0, \epsilon_0], \phi \in X} \omega_t(\phi) \) is compact in \( X \). By Theorem 2.1, for each \( \epsilon \in [0, \epsilon_0] \), \( \Phi(\epsilon, t) \) is uniformly persistent with respect to \( (X_0, \partial X_0) \), and hence, by Hale and Waltman [7, Theorem 3.2], there is a global attractor \( A_0^{\epsilon} \) for \( \Phi(\epsilon, t) : X_0 \to X_0 \). By the theorem on the uniform persistence uniform in parameters (see Smith and Zhao [12, Theorem 4.3 and Remark 4.2]), together with Claims 1 and 2 in the proof of Theorem 2.1, it then follows that there exist \( \epsilon_1 \in (0, \epsilon_0] \) and \( \beta_1 > 0 \) such that \( \liminf_{t \to +\infty} \text{dist}(\Phi(\epsilon, t)\phi, \partial X_0) \geq \beta_1 \) for all \( \phi \in X_0 \), \( \epsilon \in [0, \epsilon_1] \). Thus there exists a bounded and closed subset \( B_0^\epsilon \) of \( X_0 \) such that \( A_0^{\epsilon} \subset B_0^\epsilon \) for all \( \epsilon \in [0, \epsilon_1] \). For any \( \epsilon \in [0, \epsilon_1] \),

\[
\bigcup_{\epsilon \in [0, \epsilon_1]} \Phi(\epsilon, t)A_0^{\epsilon} = \bigcup_{\epsilon \in [0, \epsilon_1]} \Phi(\epsilon, t)B_0^\epsilon,
\]

and

\[
\bigcup_{\epsilon \in [0, \epsilon_1]} \Phi(\epsilon, t)A_0^{\epsilon} = \bigcup_{\epsilon \in [0, \epsilon_1]} A_0^{\epsilon} \subset B_0 = B_0^{\epsilon} \subset X_0.
\]

Then \( \bigcup_{\epsilon \in [0, \epsilon_1]} \Phi(\epsilon, t)A_0^{\epsilon} \) is compact in \( X_0 \). When \( \epsilon = 0 \), system (2.2) admits a unique positive equilibrium \((1 - \frac{d}{d+\lambda})N_0, N_0)\) which is globally asymptotically stable in \( X_0 \) (see Cooke et al. [1, Theorem 4.4]). By Smith and Waltman [11, Theorem 2.2] with \( U = X_0 \) and \( B = A_0^{\epsilon} \), there is an \( \tilde{\epsilon} \in (0, \epsilon_1] \) such that for each \( \epsilon \in [0, \tilde{\epsilon}] \), system (2.2) admits a positive equilibrium \((I^*(\epsilon), N^*(\epsilon))\) with \((I^*(0), N^*(0)) = ((1 - \frac{d}{d+\lambda})N_0, N_0)\), and \((I^*(\epsilon), N^*(\epsilon))\) is globally attractive in \( X_0 \).

3. DISCUSSION

Under quite general conditions, we have proved that the basic reproduction number \( R_0 \) acts as a threshold for the SIS model (1.5): when \( R_0 < 1 \), the disease dies out, and when \( R_0 > 1 \) the disease remains endemic in the sense of uniform persistence. For the latter case, if the disease induced
FIG. 1. Numerical simulations for (1.5) with $B(N) = B_1(N)$, using XPPAUT (Ermentrout [3]). The parameters are chosen as follows: $a = d = d'_1 = 1$, $b = 20$, $y = 0.5$, $T = 0.1$, and $\lambda = 20$. It is easy to verify that for these parameter values, the conditions in Theorems 2.1–2.2 are satisfied. Part (a) is for $I$ and (b) is for $N$. The initial values are $I(s) = 2$, $N(s) = 3.5$ for all $s \in [-0.1, 0]$. Note that the critical value of $\epsilon > 0$ for $R_0 > 1$ is 18.5. Convergence to the endemic equilibrium is observed for values of $\epsilon < 18.5$ ($R_0 > 1$) and is shown in the figure for $\epsilon = 5, 12, 18$. For values of $\epsilon > 18.5$ ($R_0 < 1$), convergence to the disease free equilibrium (see Theorem 2.1) is also confirmed but not given in the figure.

dehth rate $\epsilon$ is sufficiently small, we even have proved that the endemic equilibrium is globally attractive. Some numerical simulations are shown in Fig. 1. There arises a natural question: When $R_0 > 1$, does the endemic equilibrium remain globally attractive for large values of $\epsilon > 0$? In proving Theorem 2.2, we made use of a perturbation theorem established in Smith and Waltman [11, Theorem 2.2], and hence our approach fails to answer this question and the problem remains open. We hope to be able to solve this open problem by using some new technique in a forthcoming paper.
REFERENCES