Inversion of Confluent Vandermonde Matrices

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Abstract—This paper is concerned with the inversion of confluent Vandermonde matrices. A novel and simple recursive algorithm for inverting confluent Vandermonde matrices is presented. The algorithm is suitable for classroom use in both numerical as well as symbolic computation. Examples are included to illustrate the proposed algorithm. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The importance of the Vandermonde matrix is well known [1]. Inversion of this matrix is necessary in many areas of applications, such as polynomial interpolation [2,3], digital signal processing [4], and control theory [5]. For recent work of numerical treatment of Vandermonde and confluent Vandermonde matrices, see [6-9] and the references therein. In addition, a number of explicit formulas for the entries of the inverse of Vandermonde matrices have been given in [10-13]. However, an explicit recursive formula for the inversion of confluent Vandermonde matrices seems unavailable in the mathematical literature and linear algebra textbooks. The purpose of this paper is to present a novel and simple recursive algorithm for inverting Vandermonde matrices as well as confluent Vandermonde matrices, in a way more readily accessible for use in classroom and suitable for both numerical as well as symbolic computation. Part of the result has been reported in [14].

2. PRELIMINARIES AND NOTATIONS

Let m be a positive integer and let λ be a given number. For the sequence 1, (s - λ), . . . , (s - λ)m-1 of polynomials, we denote

$$s(λ, m) = [1, (s - λ), . . . , (s - λ)^{m-1}]^T.$$ 

In particular, we have

$$s(0, m) = [1, s, . . . , s^{m-1}]^T.$$ 

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Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be given pairwise distinct zeros of the polynomial

$$p(s) = (s - \lambda_1)^{n_1} \ldots (s - \lambda_r)^{n_r},$$

with $n_1 + \cdots + n_r = n$. The confluent Vandermonde matrix related to the zeros of $p(s)$ is defined to be the $n \times n$ matrix

$$V = [V_1 V_2 \ldots V_r],$$

where the block matrix $V_k = V(\lambda_k, n_k)$ is of order $n \times n_k$, having elements

$$V(\lambda_k, n_k)_{ij} = \begin{cases} (i - 1) \lambda_k^{i-j}, & \text{for } i \geq j, \\ 0, & \text{otherwise,} \end{cases}$$

for $k = 1, 2, \ldots, r$; $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, n_k$. More specifically, $V_k$ is the $n \times n_k$ matrix of coefficients that appears in the truncated Taylor expansion at $\lambda_k$, modulo $(s - \lambda_k)^{n_k}$, of $s(0, n)$. That is,

$$s(0, n) = V(\lambda_k, n_k)s(\lambda_k, n_k) \mod (s - \lambda_k)^{n_k}.$$

In the case the zeros $\lambda_1, \ldots, \lambda_r$ of $p(s)$ are simple, we have the usual Vandermonde matrix, namely,

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \lambda_1 & \cdots & \lambda_r \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n_1-1} & \lambda_2^{n_2-1} & \ldots & \lambda_r^{n_r-1} \end{bmatrix}.$$

Examples of confluent Vandermonde matrices of the types $V = [V_1(\lambda_1, 3) \ V_2(\lambda_2, 1)]$ and $V = [V_1(\lambda_1, 2) \ V_2(\lambda_2, 2)]$ can be found in Section 5.

In the sequel, we will show that the inverse of the confluent Vandermonde matrix $V$ in (1) has the form

$$V^{-1} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_r \end{bmatrix},$$

where each block matrix $W_k$ is of order $n_k \times n$, and may be computed by means of a recursive procedure.

Inversion of the usual Vandermonde matrices, as well as confluent Vandermonde matrices are in general based on polynomial interpolation [10,13]. Our present approach is based on using the Leverrier-Faddeev algorithm [3,15,16], which states that the resolvent of a given $n \times n$ matrix $A$ is given by

$$(sI - A)^{-1} = \frac{B_1 s^{n-1} + B_2 s^{n-2} + \cdots + B_n}{s^n + a_1 s^{n-1} + \cdots + a_n},$$

where $\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n$ is the characteristic polynomial of the matrix $A$, and all the $B_j$ matrices are of order $n \times n$, satisfying

$$B_1 = I, \quad a_1 = -\frac{1}{n} \text{tr}(AB_1),$$

$$B_2 = AB_1 + a_1 I, \quad a_2 = -\frac{2}{n} \text{tr}(AB_2),$$

$$\vdots$$

$$B_n = AB_{n-1} + a_{n-1} I, \quad a_n = -\frac{1}{n} \text{tr}(AB_n),$$

with $0 = AB_n + a_n I$ terminating as a check of computation. Here, $\text{tr}$ stands for the trace of a matrix.
3. MAIN RESULTS

Let $J = \text{diag}(J_1, \ldots, J_r)$ be the block diagonal matrix, where

$$J_k = J(\lambda_k, n_k) = \begin{bmatrix}
\lambda_k & 1 & 0 & \cdots & 0 \\
0 & \lambda_k & 1 & \vdots & \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & \lambda_k \\
0 & \cdots & 0 & 0 & \lambda_k
\end{bmatrix}$$

is the $n_k \times n_k$ Jordan block with eigenvalue $\lambda_k$. Then $J$ has the characteristic polynomial

$$\det(sI - J) = (s - \lambda_1)^{n_1} \cdots (s - \lambda_r)^{n_r} = p(s).$$

Substituting $A = J$ into equations (2) and (3) of the Leverrier-Faddeev algorithm, we see immediately that

$$p(s)(sI - J)^{-1} = B_1 s^{n-1} + B_2 s^{n-2} + \cdots + B_n,$$

where

$$B_1 = I,$$
$$B_2 = JB_1 + a_1 I,$$
$$\vdots$$
$$B_n = JB_{n-1} + a_{n-1} I,$$
$$0 = JB_n + a_n I.$$

$J = \text{diag}(J_1, \ldots, J_r)$ being block diagonal, so are all the $B_j$ matrices. In fact,

$$B_j = \text{diag}(B_{j,1}, B_{j,2}, \ldots, B_{j,n}), \quad j = 1, 2, \ldots, n,$$

and each block matrix $B_{j,k}$ is of order $n_k \times n_k$, satisfying

$$B_{1,k} = I_k,$$
$$B_{2,k} = J_k B_{1,k} + a_1 I_k,$$
$$\vdots$$
$$B_{n,k} = J_k B_{n-1,k} + a_{n-1} I_k,$$
$$0 = J_k B_{n,k} + a_n I_k,$$

where $I_k$ is the $n_k \times n_k$ identity matrix.

Let us now denote

$$p_k(s) = \frac{p(s)}{(s - \lambda_k)^{n_k}}, \quad k = 1, \ldots, r,$$

and the $n_k$-dimensional column vector $\theta_k = [0, \ldots, 0, 1]^T$. Then it is easy to show

$$(sI - J_k)^{-1} \theta_k = \left[ \frac{1}{(s - \lambda_k)^{n_k}}, \ldots, \frac{1}{(s - \lambda_k)^{n_k}} \right]^T.$$

If we postmultiply both sides of (4) by the $n$-dimensional column vector $\theta = [\theta_1^T, \ldots, \theta_r^T]^T$ and taking (6) into account, we easily get

$$\begin{bmatrix}
p_1(s)(\lambda_1, n_1) \\
\vdots \\
p_r(s)(\lambda_r, n_r)
\end{bmatrix} = \begin{bmatrix}
H_1 \\
\vdots \\
H_r
\end{bmatrix} s(0, n).$$
Each block matrix $H_k$ is of the form

$$H_k = \begin{bmatrix} B_{n,k} \theta_k & \ldots & B_{1,k} \theta_k \end{bmatrix},$$

having order $n_k \times n$.

Comparing in turn for $k = 1, 2, \ldots, r$, the truncated Taylor expansions at $\lambda_k$, modulo $(s - \lambda_k)^{n_k}$, of both sides in (7) and putting these results together, we get

$$\text{diag}(P_1, \ldots, P_r) = [H_1 \ldots H_r]$$

where each block $P_k$ is the $n_k \times n_k$ upper triangular matrix given by

$$P_k = p_k(J_k) = \sum_{j=0}^{n_k-1} \frac{p_k^{(j)}(\lambda_k)}{j!}(N_k)^j.$$

Note that $N_k = J(0, n_k) = \lambda_k I_k$ is nilpotent of order $n_k$.

If we can show that each $P_k$ is invertible, then

$$V^{-1} = \begin{bmatrix} P_1^{-1}H_1 \\ \vdots \\ P_r^{-1}H_r \end{bmatrix}.$$  \hfill (9)

To this end, we require the following lemma which is an easy consequence of the partial fraction expansion of $1/p(s)$ and the fact that $N_k$ is nilpotent.

**Lemma 3.1.** Let there be given the partial fraction expansion

$$\frac{1}{p(s)} = \sum_{k=1}^{r} \left( \frac{K_{k,n_k}}{(s - \lambda_k)^{n_k}} + \frac{K_{k,n_k-1}}{(s - \lambda_k)^{n_k-1}} + \cdots + \frac{K_{k,1}}{s - \lambda_k} \right).$$

Then for $k = 1, 2, \ldots, r$,

$$P_k^{-1} = \sum_{j=0}^{n_k-1} K_{k,j}(N_k)^j = \mathcal{K}_k(J_k),$$

where the polynomial $\mathcal{K}_k(s)$ is given by

$$\mathcal{K}_k(s) = K_{k,n_k} + K_{k,n_k-1}(s - \lambda_k) + \cdots + K_{k,1}(s - \lambda_k)^{n_k-1}.$$  \hfill (10)

**Proof.** Multiplying both sides of equation (10) by $p(s)$, we have

$$1 = p_1(s)\mathcal{K}_1(s) + \cdots + p_r(s)\mathcal{K}_r(s).$$ \hfill (11)

Fix $k \in \{1, \ldots, r\}$. Then for $i \neq k$,

$$p_i(s) = \frac{p(s)}{(s - \lambda_i)^{n_i}} = (s - \lambda_k)^{n_k} g_i(s),$$

for some polynomials $g_i(s)$, and it follows from $(J_k - \lambda_k I)^{n_k} = (N_k)^{n_k} = 0$ that $p_i(J_k) = 0$ for $i \neq k$. Thus, setting $s = J_k$ in (11) yields

$$I = p_k(J_k)\mathcal{K}_k(J_k),$$

and so $P_k = p_k(J_k)$ is invertible with inverse $P_k^{-1} = \mathcal{K}_k(J_k)$.

Putting the above results together with equations (8) and (9), we readily obtain the following main result.
**Theorem 3.1.** The inverse of \( V = [V_1(\lambda_1, n_1)V_2(\lambda_2, n_2) \ldots V_r(\lambda_r, n_r)] \) related to the pairwise distinct zeros \( \lambda_1, \ldots, \lambda_r \) of \( p(s) \) is given by

\[
V^{-1} = \begin{bmatrix}
W_1 \\
W_2 \\
\vdots \\
W_r
\end{bmatrix},
\]

where each block matrix \( W_k = W(\lambda_k, n_k) - [K_k(J_k)B_n, k\theta_k \; K_k(J_k)B_{n-1}, k\theta_k \; \ldots \; K_k(J_k)B_1, k\theta_k] \) is of order \( n_k \times n \).

Taking (5) into account, we find that

\[
\mathcal{K}_k(J_k)B_{j,k} = B_{j,k}\mathcal{K}_k(J_k), \quad j = 1, 2, \ldots, n,
\]

so that \( B_{1,k}\mathcal{K}_k(J_k) = \mathcal{K}_k(J_k), \) and for \( j = 2, \ldots, n, \)

\[
B_{j,k}\mathcal{K}_k(J_k) = J_kB_{j-1,k}\mathcal{K}_k(J_k) + a_{j-1}\mathcal{K}_k(J_k)
\]

\[
= (\lambda_k I_k + N_k)B_{j-1,k}\mathcal{K}_k(J_k) + a_{j-1}\mathcal{K}_k(J_k).
\]

Moreover,

\[
(\lambda_k I_k + N_k)B_{n,k}\mathcal{K}_k(J_k) + a_n\mathcal{K}_k(J_k) = 0.
\]

Thus, we have the following theorem.

**Theorem 3.2.** The column vectors \( h_i \) of the block matrix \( W_k = [h_{n_1}, \ldots, h_1] \) in \( V^{-1} = [W_1, \ldots, W_r]^T \) may be recursively computed by the following scheme:

\[
h_1 = \mathcal{K}_k(J_k)\theta_k
g_k = K_k(n_k, n_k, \ldots, n_k)\theta_k + \cdots + K_k(1)(n_k)^{n_k - 1}\theta_k,
\]

\[
h_2 = (\lambda_k I_k + N_k) h_1 + a_1 h_1,
\]

\[
\vdots
\]

\[
h_{n-1} = (\lambda_k I_k + N_k) h_{n-2} + a_{n-2} h_1,
\]

\[
h_n = (\lambda_k I_k + N_k) h_{n-1} + a_{n-1} h_1,
\]

terminating at

\[
0 = (\lambda_k I_k + N_k) h_n + a_n h_1.
\]

Thus, we note that all these vectors \( h_i = [h_{i,1}, \ldots, h_{i,n_i}]^T \) can be represented in the form

\[
h_i = h_{i, n_i} + h_{i, n_i - 1}s + \cdots + h_{i, 1}s^{n_i - 1}|_{s = \mathcal{N}_0} \theta_k.
\]  \hspace{1cm} (12)

\section*{4. Algorithm}

Based on the result given in the previous section and noting the facts that \( N_k \) is nilpotent of order \( n_k \) and the vectors \( h_i \) have the representation (12), we are led to the following recursive algorithm for determining the inverse of confluent Vandermonde matrices.

**Algorithm for inverting confluent Vandermonde matrices.**

Let \( \lambda_1, \lambda_2, \ldots, \lambda_r \) be pairwise distinct zeros of the polynomial

\[
p(s) = (s - \lambda_1)^{n_1} \cdots (s - \lambda_r)^{n_r}
\]

\[
= s^n + a_1 s^{n-1} + \cdots + a_n,
\]
given together with the partial fraction expansion of
\[
\frac{1}{p(s)} = \sum_{k=1}^{r} \left( \frac{K_{k,n_k}}{(s - \lambda_k)^{n_k}} + \frac{K_{k,n_k-1}}{(s - \lambda_k)^{n_k-1}} + \cdots + \frac{K_{k,1}}{s - \lambda_k} \right).
\]

For each \( k \in \{1, 2, \ldots, r\} \), compute recursively polynomials
\[
h_1(s), h_2(s), \ldots, h_n(s),
\]
all of degree at most \( n_k - 1 \) by means of the following scheme:
\[
h_1(s) = K_{k,n_k} + sK_{k,n_k-1} + \cdots + s^{n_k-1}K_{k,1},
\]
\[
h_2(s) = (\lambda_k + s)h_1(s) + a_1h_1(s) \mod s^{n_k},
\]
\[
h_3(s) = (\lambda_k + s)h_2(s) + a_2h_1(s) \mod s^{n_k},
\]
\[
\vdots
\]
\[
h_n(s) = (\lambda_k + s)h_{n-1}(s) + a_{n-1}h_1(s) \mod s^{n_k},
\]
terminating at
\[
0 = (\lambda_k + s)h_n(s) + a_nh_1(s) \mod s^{n_k}.
\]
Obtain the block matrix \( W_k = W(\lambda_k, n_k) \) of order \( n_k \times n \) via the equality
\[
\begin{bmatrix}
1 & s^{n_k-1} & \cdots & s
\end{bmatrix} W(\lambda_k, n_k) = [h_n(s) \ h_{n-1}(s) \ \cdots \ h_1(s)].
\]
The inverse of the confluent Vandermonde matrix
\[
V = [V(\lambda_1, n_1) V(\lambda_2, n_2) \cdots V(\lambda_r, n_r)]
\]
related to the pairwise distinct zeros \( \lambda_1, \lambda_2, \ldots, \lambda_r \) of \( p(s) \) may then be obtained by setting
\[
V^{-1} = \begin{bmatrix}
W(\lambda_1, n_1) \\
W(\lambda_2, n_2) \\
\vdots \\
W(\lambda_r, n_r)
\end{bmatrix}.
\]

REMARK. It is noted that a check on the accuracy of the computation of polynomials \( h_1(s), \ldots, h_n(s) \) is provided by the last polynomial \( (\lambda_k + s)h_n(s) + a_nh_1(s) \), which should result identically in the zero polynomial 0 when modulo \( s^{n_k} \) is performed.

5. EXAMPLES

The following examples will serve to illustrate the recursive algorithm as presented in Section 4.

EXAMPLE 1. Let the confluent Vandermonde matrix \( V \) in (1) be given by
\[
V = \begin{bmatrix}
1 & 0 & 0 & 1 \\
\lambda_1 & 1 & 0 & \lambda_2 \\
\lambda_1^2 & 2\lambda_1 & 1 & \lambda_2^2 \\
\lambda_1^3 & 3\lambda_1^2 & 3\lambda_1 & \lambda_2^3
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 1 \\
-2 & 1 & 0 & 3 \\
4 & -4 & 1 & 9 \\
-8 & 12 & -6 & 27
\end{bmatrix},
\]
for which \( \lambda_1 = -2, n_1 = 3, \) and \( \lambda_2 = 3, n_2 = 1. \) The coefficients of the polynomial \( p(s) = (s + 2)^3(s - 3) \) are given by \( a_1 = 3, a_2 = -6, a_3 = -28, \) and \( a_4 = -24. \) It is easy to determine the partial fraction expansion of \( 1/p(s) \) to be
\[
\frac{1}{(s + 2)^3(s - 3)} = \frac{-1}{5} + \frac{-1}{25} \frac{1}{s + 2} + \frac{-1}{125} \frac{1}{s + 2} + \frac{1}{125} \frac{1}{s - 3}.
\]
Consider first the case $\lambda_1 = -2$. Clearly,

$$h_1(s) = -\frac{1}{5} - \frac{s}{25} - \frac{s^2}{125}.$$  

Then

$$h_2(s) = (-2 + s)h_1(s) + 3h_1(s) \mod s^3 = -\frac{1}{5} - \frac{6s}{25} - \frac{6s^2}{125},$$

$$h_3(s) = (-2 + s)h_2(s) - 6h_1(s) \mod s^3 = \frac{8}{5} + \frac{13s}{25} - \frac{12s^2}{125},$$

$$h_4(s) = (-2 + s)h_3(s) - 28h_1(s) \mod s^3 = \frac{12}{5} + \frac{42s}{25} + \frac{117s^2}{125}.$$  

As a check of computation, we verify that

$$(-2 + s)h_4(s) - 24h_1(s) \mod s^3 = \frac{117s^3}{125} \mod s^3 = 0.$$  

Thus, it follows from $[s^2 \ s \ 1]W_1 = [h_4(s) \ h_3(s) \ h_2(s) \ h_1(s)]$ that

$$W_1 = \begin{bmatrix} 117 & -12 & 6 & 1 \\ 125 & 125 & 125 & 125 \\ 42 & 13 & 6 & 1 \\ 25 & 25 & 25 & 25 \\ 12 & 8 & 1 & 1 \\ 5 & 5 & 5 & 5 \end{bmatrix}.$$  

Similarly, for $\lambda_2 = 3$, we find the corresponding polynomials $h_1(s), \ldots, h_4(s)$ to be

$$h_1(s) = \frac{1}{125},$$

$$h_2(s) = (3 + s)h_1(s) + 3h_1(s) \mod s = -\frac{6}{125},$$

$$h_3(s) = (3 + s)h_2(s) - 6h_1(s) \mod s = \frac{12}{125},$$

$$h_4(s) = (3 + s)h_3(s) - 28h_1(s) \mod s = \frac{8}{125},$$

and

$$(3 + s)h_4(s) - 24h_1(s) \mod s = 3 \cdot \frac{8}{125} - \frac{24}{125} = 0.$$  

Then $[1]W_2 = [h_4(s) \ h_3(s) \ h_2(s) \ h_1(s)]$ gives

$$W_2 = \begin{bmatrix} 8 & 12 & 6 & 1 \\ 125 & 125 & 125 & 125 \end{bmatrix}.$$  

Finally, we have

$$V^{-1} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} 117 & -12 & 6 & 1 \\ 125 & 125 & 125 & 125 \\ 42 & 13 & 6 & 1 \\ 25 & 25 & 25 & 25 \\ 12 & 8 & 1 & 1 \\ 5 & 5 & 5 & 5 \\ 8 & 12 & 6 & 1 \\ 125 & 125 & 125 & 125 \end{bmatrix}.$$
EXAMPLE 2. We now apply the recursive algorithm to derive an explicit formula for the inverse of the confluent Vandermonde matrix

\[
V = \begin{bmatrix}
V_1(\lambda_1, 2) & V_2(\lambda_2, 2)
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 1 & 0 \\
\lambda_1 & 1 & \lambda_2 & 1 \\
2\lambda_1 & \lambda_1^2 & 2\lambda_2 & \lambda_2^2 \\
3\lambda_1^2 & 3\lambda_1^3 & 3\lambda_2^2 & 3\lambda_2^3
\end{bmatrix},
\]

where \( \lambda_1 = a \) and \( \lambda_2 = b \) are pairwise distinct zeros of the polynomial

\[
p(s) = (s - a)^2(s - b)^2
= s^4 - 2(a + b)s^3 + (a^2 + 4ab + b^2)s^2 - 2ab(a + b)s + a^2b^2.
\]

From the partial fraction expansion

\[
\frac{1}{(s - a)^2(s - b)^2} = \frac{1}{(s - a)^2} + \frac{2}{(s - a)^3} + \frac{1}{(s - b)^2} + \frac{2}{(s - b)^3},
\]

we find, for the case \( \lambda_2 = b \), the corresponding polynomials \( h_1(s), \ldots, h_4(s) \) as follows:

\[
h_1(s) = \frac{1}{(a - b)^2} + \frac{2s}{(a - b)^3},
\]

\[
h_2(s) = (b + s)h_1(s) - 2(a + b)h_1(s) \mod s^2
= \frac{-2a + b}{(a - b)^2} + \frac{3(a + b)s}{(a - b)^3},
\]

\[
h_3(s) = (b + s)h_2(s) + (a^2 + 4ab + b^2)h_1(s) \mod s^2
= \frac{a(a + 2b)}{(a - b)^2} + \frac{6abs}{(a - b)^3},
\]

\[
h_4(s) = (b + s)h_3(s) - 2ab(a + b)h_1(s) \mod s^2
= \frac{-a^2b}{(a - b)^2} + \frac{a^2(a - 3b)s}{(a - b)^3},
\]

so that

\[
W_2 = \begin{bmatrix}
\frac{a^2(a - 3b)}{(a - b)^3} & \frac{6ab}{(a - b)^3} & \frac{-3(a + b)}{(a - b)^3} & \frac{2}{(a - b)^3} \\
\frac{-a^2b}{(a - b)^2} & \frac{a(a + 2b)}{(a - b)^2} & \frac{-2(a + b)}{(a - b)^2} & \frac{1}{(a - b)^2} \\
\end{bmatrix}.
\]

The block matrix \( W_1 \) associated with \( \lambda_1 = a \) can be obtained from \( W_2 \) by simply interchanging \( a \) and \( b \), so that

\[
W_1 = \begin{bmatrix}
\frac{b^2(b - 3a)}{(a - b)^3} & \frac{6ab}{(b - a)^3} & \frac{-3(a + b)}{(b - a)^3} & \frac{2}{(b - a)^3} \\
\frac{-b^2a}{(b - a)^2} & \frac{b(b + 2a)}{(b - a)^2} & \frac{-2b + a}{(b - a)^2} & \frac{1}{(b - a)^2} \\
\end{bmatrix}.
\]

Hence,

\[
V^{-1} = \begin{bmatrix}
W_1 & W_2
\end{bmatrix} = \begin{bmatrix}
\frac{b^2(b - 3a)}{(a - b)^3} & \frac{6ab}{(b - a)^3} & \frac{-3(a + b)}{(b - a)^3} & \frac{2}{(b - a)^3} \\
\frac{b^2a}{(a - b)^2} & \frac{b(b + 2a)}{(a - b)^2} & \frac{-2b + a}{(a - b)^2} & \frac{1}{(a - b)^2} \\
\frac{a^2(a - 3b)}{(a - b)^3} & \frac{6ab}{(a - b)^3} & \frac{-3(a + b)}{(a - b)^3} & \frac{2}{(a - b)^3} \\
\frac{-a^2b}{(a - b)^2} & \frac{a(a + 2b)}{(a - b)^2} & \frac{-2(a + b)}{(a - b)^2} & \frac{1}{(a - b)^2}
\end{bmatrix}.
\]
6. CONCLUSION

A simple and novel recursive algorithm was presented for determining the inverse of confluent Vandermonde matrices. The proposed algorithm essentially computes column by column, one block of the inverse confluent Vandermonde matrix at a time. It can be used to derive an explicit formula of the inverse of confluent Vandermonde matrices.

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