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Polynomial Algebra for Birkhoff Interpolants

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Abstract We introduce a unifying formulation of a number of related problems which can all be solved using a contour integral formula. Each of these problems requires finding a non-trivial linear combination of some of the values of a function f, and its first and higher derivatives, at a number of data points. This linear combination is required to have zero value when f is a polynomial of up to a specific degree p. Examples of this type of problem include Lagrange, Hermite and Hermite-Birkhoff interpolation; and various numerical quadrature and differentiation formulae. Other applications include the estimation of missing data and root-finding.

Keywords Lagrange, Hermite, and Hermite-Birkhoff interpolation · Contour integrals · Barycentric form · Root-finding

Mathematics Subject Classification (2000) 41A05 · 65D05 · 65D25 · 65D30

1 Introduction

The purpose of this paper is to present a number of approximation formulae using a single standard formulation. These include formulae for Lagrange, Hermite and Hermite–Birkhoff interpolation, divided difference formulae, numerical quadrature and numerical differentiation. Each of these approximation formulae can be written as a non-trivial linear combination

$$\sum_{i=1}^{n} \sum_{j\in S_i} \widehat{a}_{ij} f^{(j)}(\tau_i) = 0, \tag{1}$$

where f is a polynomial of degree not exceeding p, and S_i is a subset of $\{0, 1, \ldots, s_i - 1\}$ such that $s_i - 1 \in S_i$. The value of p is $-2 + \sum_{i=1}^{n} |S_i|$, where $|S_i|$ is the cardinality of S_i.

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If $S_i = \{0, 1, \dots, s_i - 1\}$, for each $i = 1, 2, \dots, n$, we will rewrite (1) in the form

$$\sum_{i=1}^{n} \sum_{j=0}^{s_i - 1} a_{ij} f^{(j)}(\tau_i) = 0. \tag{2}$$

The common feature of these formulations is their relation to contour integrals of the form

$$\frac{1}{2\pi i} \oint_C R(z) f(z) dz, \tag{3}$$

where R is a rational function related to the problem. By using the Cauchy integral formula we will find the approximation we require.

The paper is organized as follows: a review of known literature on Hermite–Birkhoff interpolation and related problems will be presented in Section 2. This is followed by Section 3, where the central result is given and by Section 4 where diverse applications are considered. Results and discussion on the existence and other properties of these approximations are presented in Section 5.

2 Literature review

3 Solution by contour integrals

3.1 Statement of the problem

We are concerned with a function f and various approximations related to it. The approximations will have order p, in the sense that if f is replaced by a polynomial of degree p, then the approximation will be exact. The approximations will all have the form

$$\sum_{i=1}^{n} \sum_{j=0}^{s_i - 1} a_{ij} f^{(j)}(\tau_i) = 0, \quad \deg(f) \le p. \tag{4}$$

Because of its relation to Hermite interpolation, we will refer to the construction of approximations satisfying (4) as the "Hermite case". In Subsection 3.2 we will obtain formulae for the coefficients a_{ij}.

A more general problem, which we will refer to as the "Birkhoff case", is to assume that there is missing data. In other words, we will assume that for each i, $f^{(j)}(\tau_i)$ is given only for a subset S_i of $\{0, 1, \dots, s_i - 1\}$. We will assume that such a subset always contains $s_i - 1$. This will enable us to avoid the situation that s_i could have been replaced by a lower integer. With this more general case, (4) will be replaced by

$$\sum_{i=1}^{n} \sum_{j \in S_i} \widehat{a}_{ij} f^{(j)}(\tau_i) = 0, \quad \deg(f) \le p. \tag{5}$$

By counting the number of constraints, we would hope to be able to solve (4) with $p = -2 + \sum_{i=1}^{n} s_i$. It will be shown by construction that this is always possible. In the Birkhoff case, again by counting constraints, we see that it is appropriate to aim for and order

$$p = -2 + \sum_{i=1}^{n} |S_i|. \tag{6}$$

Conditions under which (5) is guaranteed to have a solution will be discussed in Section 5.

3.2 Solution using contour integrals in the Hermite case

Given an approximation problem in the Hermite case, we define $w(z)$ by

$$w(z) = \prod_{i=1}^{n} (z - \tau_i)^{s_i}, \tag{7}$$

where the τ_i and s_i are as in (4). We have the following lemma.

Lemma 1 *Let f be a polynomial of degree not exceeding $p = -2 + \sum_{i=1}^{n} s_i$, then*

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{w(z)} dz = 0, \tag{8}$$

where $w(z)$ is defined by (7) and the contour C is a circle with centre 0 and radius R greater than $\max_{i=1}^{n} |\tau_i|$.

Proof The proof is a classic result in Complex Analysis and can be found in [1](see Theorem 8.1, page 233). □

We can now write $w(z)^{-1}$ in (8) in partial fractions

$$\frac{1}{w(z)} = \sum_{i=1}^{n} \sum_{j=0}^{s_i-1} \frac{\gamma_{ij}}{(z - \tau_i)^{j+1}}, \tag{9}$$

so that substituting (9) into (8) and evaluating term by term, we find the result

$$\sum_{i=1}^{n} \sum_{j=0}^{s_i-1} \frac{\gamma_{ij}}{j!} f^{(j)}(\tau_i) = 0. \tag{10}$$

Hence, we can solve (4) using

$$a_{ij} = \frac{\gamma_{ij}}{j!}. \tag{11}$$

To evaluate the a_{ij} in a convenient form, use the following result

Lemma 2 *Let K denote a finite set of integers, and define $\chi_1, \chi_2, \ldots$ by the formula*

$$\chi_0 + \chi_1 t + \chi_2 t^2 + \ldots = \chi_0 \prod_{k \in K} (1 - t\theta_k)^{-s_k}\,, \tag{12}$$

where χ_0 is given. Then, $\chi_1, \chi_2, \ldots$ satisfy the recursion

$$\chi_j = \frac{1}{j} \sum_{\ell=0}^{j-1} \chi_\ell \phi_{j-\ell}\,, \tag{13}$$

where $\phi_j = \sum_{k \in K} s_k \theta_k^j$.

Proof From the logarithmic series we have

$$\ln(1-\theta_n t) = -\sum_{j=1}^{\infty} \frac{t^j}{j}\theta_k^j\,,. \tag{14}$$

For $|t|$ sufficiently small, it follows by multiplying by $-s_k$ and summing over $k \in K$, that

$$\ln\Big(\prod_{k\in K}(1-t\theta_k)^{-s_k}\Big) = \sum_{j=1}^{\infty}\frac{t^j}{j}\phi_j\,, \tag{15}$$

and therefore

$$\chi_0 + \chi_1 t + \chi_2 t^2 + \cdots = \chi_0 \exp\left(\sum_{j=1}^{\infty}\frac{t^j}{j}\phi_j\right). \tag{16}$$

Differentiate (16) with respect to t and it follows that

$$\chi_1 t + 2\chi_2 t + 3\chi_3 t^2 + \cdots = (\phi_1 + \phi_2 t + \cdots)(\chi_0 + \chi_1 t + \chi_2 t^2 + \cdots)\,. \tag{17}$$

Equate the coefficients of t^{j-1} and (13) follows. □

In our application of this lemma, let

$$K = \{1,2,\ldots,n\}\setminus\{i\},\quad \theta_k = \frac{1}{\tau_k - \tau_i},\quad \chi_0 = \frac{1}{\prod_{k\in K}(\tau_i-\tau_k)},\quad t = z - \tau_i.$$

For a given $i \in \{1,2,\ldots,n\}$, evaluate the Laurent expansion about $z = \tau_i$ as follows

$$\begin{aligned}\frac{1}{w(z)} &= \frac{1}{(z-\tau_i)^{s_i}}\prod_{k\in K}(z-\tau_k)^{-s_k}\\ &= \frac{1}{(z-\tau_i)^{s_i}}\prod_{k\in K}\Big((\tau_i-\tau_k)\Big(1-\frac{z-\tau_i}{\tau_k-\tau_i}\Big)\Big)^{-s_k}\\ &= \frac{1}{(z-\tau_i)^{s_i}}\chi_0\prod_{k\in K}(1-t\theta_k)^{-s_k}\\ &= \frac{1}{(z-\tau_i)^{s_i}}(\chi_0 + \chi_1(z-\tau_i) + \chi_2(z-\tau_1)^2 + \cdots)\end{aligned} \tag{18}$$

and hence

$$\gamma_{ij} = \chi_{s_i-j-1},\quad a_{ij} = \frac{\chi_{s_i-j-1}}{j!}. \tag{19}$$

Having found the coefficients in (4), we generalize to the Birkhoff case.

3.3 Solution in the Birkhoff case

To allow for missing data, replace (8) by

$$\frac{1}{2\pi i}\oint_C \frac{B(z)f(z)}{w(z)}dz = 0, \tag{20}$$

where

$$B(z) = b_0 + b_1 z + \cdots + b_m z^m \tag{21}$$

and m is the number of missing data items. The integral is still zero if f is now restricted to a polynomial of degree

$$p = -2 - m + \sum_{i=1}^{n} s_i = -2 + \sum_{i=1}^{n} |S_i|. \tag{22}$$

The coefficients in $B(z)$, suitably normalized, are to be chosen so that, in the partial fraction expansion of $B(z)/w(z)$, which we will write as

$$\frac{B(z)}{w(z)} = \sum_{i=1}^{n}\sum_{j=0}^{s_i-1} \frac{\widehat{\gamma}_{ij}}{(z-\tau_i)^{j+1}}, \tag{23}$$

the value of $\widehat{\gamma}_{ij}$ will vanish whenever $j \notin S_i$. To solve (5), it is then possible to use a slightly modified form of Lemma 1 to give the result

$$\widehat{a}_{ij} = \frac{\widehat{\gamma}_{ij}}{j!}, \quad j \in S_i. \tag{24}$$

Remark 1 The partial fraction expansion of (23) can be represented in a matrix notation. To obtain the corresponding matrix notation we recall that in the Hermite case if the local series in (**??**):

$$B(z) = \beta_{i,0} + \beta_{i,1}(z-\tau_i) + \ldots + \beta_{i,s_i-1}(z-\tau_i)^{s_i-1} + O((z-\tau_i)^{s_i})\,, \tag{25}$$

We have

$$\frac{B(z)}{w(z)} = \sum_{i=1}^{n}\sum_{j=0}^{s_i-1}\sum_{k=0}^{j} \gamma_{ij}\beta_{ik}(z-\tau_i)^{k-j-1} \tag{26}$$

interchanging orders of summation as in Lemma **??**

$$= \sum_{i=1}^{n}\sum_{j=0}^{s_i-1} \widehat{\gamma}_{ij}(z-\tau_i)^{(-j-1)}\,, \tag{27}$$

where

$$\widehat{\gamma} = \sum_{k=0}^{s_i-1-j} \gamma_{i,j+k}\beta_{ik}\,. \tag{28}$$

Interpreting this in matrix form, we have

$$\mathbf{\Gamma}_i \mathbf{T}_i \mathbf{b} = \begin{bmatrix} \widehat{\gamma}_{i,0} \\ \widehat{\gamma}_{i,1} \\ \vdots \\ \widehat{\gamma}_{i,s_i-1} \end{bmatrix} \tag{29}$$

where

$$\boldsymbol{\Gamma}_i = \begin{bmatrix} \gamma_{i,0} & \gamma_{i,1} & \cdots & \gamma_{i,s_i-2} & \gamma_{i,s_i-1} \\ \gamma_{i,1} & \gamma_{i,2} & \cdots & \gamma_{i,s_i-1} & 0 \\ \vdots & & & & \vdots \\ \gamma_{i,s_i-1} & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{30}$$

Note that for $0 \leqslant j \leqslant s_i - 1$ and $0 \leqslant k \leqslant s_i - 1$, the (j, k) entry (indexing from 0) of $\boldsymbol{\Gamma}_i$ is

$$\boldsymbol{\Gamma}_i(j,k) = \begin{cases} \gamma_{i,k+j} & \text{if } k + j \leqslant s_i - 1 \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

and

$$\mathbf{T}_i = \begin{bmatrix} 1 & \tau_i & \tau_i^2 & \dots & \tau_i^m \\ & 1 & 2\tau_i & \dots & m\tau_i^{m-1} \\ & & 1 & \dots & \frac{m(m-1)}{2!}\tau_i^{m-2} \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}, \tag{32}$$

Similarly,

$$\mathbf{T}_i(j,k) = \binom{k}{j} t_i^{k-j} \tag{33}$$

(Recall that $\binom{k}{j} = 0$, if $j > k$).

Use Rob's handwritten notes to clarify the following theorem a bit more √

Theorem 1 *For $i = 1, 2, \dots, n$, let $\mathbf{M}_i$ denote the $s_i \times (m+1)$ matrix*

$$\mathbf{M}_i = \boldsymbol{\Gamma}_i \mathbf{T}_i, \tag{34}$$

where $\boldsymbol{\Gamma}_i$ is the $s_i \times (m+1)$ matrix

$$\boldsymbol{\Gamma}_i = \begin{bmatrix} \gamma_{i,0} & \gamma_{i,1} & \cdots & \gamma_{i,s_i-2} & \gamma_{i,s_i-1} \\ \gamma_{i,1} & \gamma_{i,2} & \cdots & \gamma_{i,s_i-1} & 0 \\ \vdots & & & & \vdots \\ \gamma_{i,s_i-1} & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{35}$$

where for $0 \leqslant j \leqslant s_i - 1$ and $0 \leqslant k \leqslant s_i - 1$, the (j, k) entry (indexing from 0) of $\boldsymbol{\Gamma}_i$ is

$$\boldsymbol{\Gamma}_i(j,k) = \begin{cases} \gamma_{i,k+j} & \textit{if } k + j \leqslant s_i - 1 \\ 0 & \textit{otherwise} \end{cases} \tag{36}$$

and $\mathbf{T}_i$ is the $(m+1) \times (m+1)$ matrix

$$\mathbf{T}_i = \begin{bmatrix} 1 & \tau_i & \tau_i^2 & \dots & \tau_i^m \\ & 1 & 2\tau_i & \dots & m\tau_i^{m-1} \\ & & 1 & \dots & \frac{m(m-1)}{2!}\tau_i^{m-2} \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}, \tag{37}$$

with (j,k) element

$$\mathbf{T}_i(j,k) = \binom{k}{j} t_i^{k-j} \tag{38}$$

(Recall that $\binom{k}{j} = 0$*, if* $j > k$*). Let* $\widetilde{\mathbf{M}}$ *denote the* $m \times (m+1)$ *matrix*

$$\widetilde{\mathbf{M}} = \begin{bmatrix} \widetilde{M}_1 \\ \widetilde{M}_2 \\ \vdots \\ \widetilde{M}_n \end{bmatrix}, \tag{39}$$

where $\widetilde{\mathbf{M}}_i$ *is made up from rows of* $\mathbf{M}_i$ *such that row number* j *is included if and only if* $j \notin S_i$*.*

Then, if $\widetilde{\mathbf{M}}$ *has rank* m*, the coefficients in* (21) *are a non-zero solution of*

$$\widetilde{\mathbf{M}}\mathbf{b} = 0\,, \quad \textit{where} \quad \mathbf{b} = \begin{bmatrix} b_0 \; b_1 \; \dots \; b_m \end{bmatrix}^T. \tag{40}$$

Proof The proof requires finding m homogeneous linear equations in the elements of $\mathbf{b}$. This, in turn, requires finding the values of $\widetilde{\gamma}_{ij}$ in (24), in terms of $\mathbf{b}$. The first step is to write the Taylor coefficients about τ_i in terms of $\mathbf{b}$. This is given by the formula

$$\begin{bmatrix} B(\tau_i) \\ B'(\tau_i) \\ \frac{1}{2!}B''(\tau_i) \\ \vdots \\ \frac{1}{m!}B^{(m)}(\tau_i) \end{bmatrix} = \mathbf{T}_i\mathbf{b}. \tag{41}$$

The values of $\widehat{\gamma}_{ij}$, for $j = 0, 1, \dots, s_i - 1$, are found by multiplying (9) by

$$B(z) = B(\tau_i) + B'(\tau_i)(z - \tau_i) + \dots + \frac{1}{m!}B^{(m)}(\tau_i)(z - \tau_i)^m. \tag{42}$$

This gives

$$\widehat{\gamma}_{ij} = \sum_{\ell=0}^{\min(j,m)} \gamma_{i\ell} \frac{B^{(j-\ell)}(\tau_i)}{(j-\ell)!}, \tag{43}$$

which can be written as

$$\begin{bmatrix} \widehat{\gamma}_{i0} \\ \widehat{\gamma}_{i1} \\ \vdots \\ \widehat{\gamma}_{i,s_i-1} \end{bmatrix} = \mathbf{\Gamma}_i \begin{bmatrix} B(\tau_i) \\ B'(\tau_i) \\ \frac{1}{2!}B''(\tau_i) \\ \vdots \\ \frac{1}{m!}B^{(m)}(\tau_i) \end{bmatrix}. \tag{44}$$

Substitute from (41) and use (34); it is found that

$$\begin{bmatrix} \widehat{\gamma}_{i0} \\ \widehat{\gamma}_{i1} \\ \vdots \\ \widehat{\gamma}_{i,s_i-1} \end{bmatrix} = \mathbf{\Gamma}_i(\mathbf{T}_i\mathbf{b}) = \mathbf{M}_i\mathbf{b} \tag{45}$$

and because $\widehat{\gamma}_{ij} = 0$ for $j \notin S_i$, it follows that $\widetilde{\mathbf{M}}_i\mathbf{b} = 0$. Combine this result for all $i = 1, 2, \dots, n$ and the theorem follows. □

As in Subsection 3.2. we conclude this investigation of the Birkhoff case by finding the coefficients in(5). These are

$$\widehat{a}_{ij} = \frac{\widehat{\gamma}_{ij}}{l!}, \qquad i \in \{1, 2, \ldots, n\}, j \in S_i. \tag{46}$$

4 Specific applications

4.1 Interpolation

Given data points as in Section 3, we can obtain a formula for the interpolational polynomial by adding to the set $\{\tau_1, \tau_2, \ldots, \tau_n\}$ the additional point $\tau_0 = t$, with $s_0 = 1$. This means, in the Hermite case, that we make use of the contour integral (8) in which $w(z)$ is replaced by

$$w(z) = \prod_{i=0}^{n} (z - \tau_i)^{s_i} = (z - t) \prod_{i=1}^{n} (z - \tau_i)^{s_i} \tag{47}$$

Following through with the construction of Subsection 3.2, we find the modified form of (4) as follows:

$$a_{00} f(t) + \sum_{i=1}^{n} \sum_{j=0}^{s_i - 1} a_{ij} f^{(j)}(\tau_i) = 0, \quad \deg(f) \leq -1 + \sum_{i=1}^{n} s_i. \tag{48}$$

Assuming that $a_{00} \neq 0$, this gives the interpolation formula

$$f(t) = \sum_{i=1}^{n} \sum_{j=0}^{s_i - 1} \frac{-a_{ij}}{a_{00}} f^{(j)}(\tau_i), \quad \deg(f) \leq -1 + \sum_{i=1}^{n} s_i. \tag{49}$$

In the Birkhoff case, we can do the construction described in Subsection 3.3, again with $w(z)$ replaced by (47). This gives a formula

$$f(t) = \sum_{i=1}^{n} \sum_{j \in S_i} \frac{-\widehat{a}_{ij}}{\widehat{a}_{00}} f^{(j)}(\tau_i), \quad \deg(f) \leq -1 + \sum_{i=1}^{n} |S_i|, \tag{50}$$

again assuming that $\widehat{a}_{00} \neq 0$.

4.2 Filling in missing data

We discuss the problem of inserting some or all of the data which was not originally available in an instance of the Birkhoff case. The inserted data is to be calculated from the interpolational polynomial defined in terms of the original Hermite-Birkhoff data. However, this problem can also be looked at in terms of the zero-valued linear combination of the original data and a single additional piece of data, which is to hold for all polynomials of up to degree $-1 + \sum_{i=1}^{n} |S_i|$. We will describe this construction and then establish as a main result that these two ways of looking at the data-filling problem are identical.

Example 1 In the case $n = 3$ with $\tau_1 = 1$, $\tau_2 = 2$, $\tau_3 = 4$ and $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{1\}$ suppose we wish to estimate $f(\tau_2)$. If the missing point is inserted we find the confluent divided difference table

$$\begin{array}{c|lllll}
1 & f_1 & & & \\
 & & f_2 - f_1 & & \\
2 & f_2 & & f_2' - f_2 + f_1 & \\
 & & f_2' & & -\frac{1}{2}f_2' + \frac{1}{12}f_3 + \frac{1}{4}f_2 - \frac{1}{3}f_1 \\
2 & f_2 & & \frac{1}{4}(f_3 - f_2) - \frac{1}{2}f_2' & \\
 & & \frac{1}{2}(f_3 - f_2) & & \\
4 & f_3 & & &
\end{array}$$

If the data, including the new point, is to fit a quadratic polynomial, then the third order confluent divided difference will be zero. Hence, $-\frac{1}{2}f_2' + \frac{1}{12}f_3 + \frac{1}{4}f_2 - \frac{1}{3}f_1 = 0$ so that

$$f_2 = \tfrac{4}{3}f_1 + 2f_2' - \tfrac{1}{3}f_3. \tag{51}$$

A similar result can be found using $w(z) = (z-1)(z-2)^2(z-4)$, $B(z) = 1$. It is found that $w(z)^{-1}$ has the partial fraction expansion

$$\frac{1}{(z-1)(z-2)^2(z-4)} = -\frac{1}{3(z-1)} + \frac{1}{4(z-2)} - \frac{1}{2(z-2)^2} + \frac{1}{12(z-4)} \tag{52}$$

For f a polynomial of degree not exceeding 2 we now have

$$0 = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{w(z)} = -\frac{1}{3}f(1) + \frac{1}{4}f(2) - \frac{1}{2}f'(2) + \frac{1}{12}f(4) \tag{53}$$

which is consistent with (51).

The general approach, which will work even if the data, including the point to be filled in, is not Hermite data, is to add a point to S_I, for some $I \in \{1, 2, \ldots, n\}$. Denote this additional point by J, where $J \notin S_I$. Now carry out the construction in Subsection 3.3 with S_I replaced by $S_I \cup \{J\}$ to obtain a result

$$a_{IJ}f^{(J)}(\tau_I) + \sum_{i=1}^{n}\sum_{j\in S_i} a_{ij}f^{(j)}(\tau_i) = 0 \tag{54}$$

The connection between the interpolational polynomial and the fill-in result given by (54) is as follows.

Theorem 2 *If a_{IJ} in* (54) *is not zero, and $\varphi(t)$ is the interpolational polynomial based on data $f^{(j)}(\tau_i)$, $j \in S_i$, $i \in \{1, 2, \ldots, n\}$, then*

$$f^{(J)}(\tau_I) = \varphi^{(J)}(\tau_I). \tag{55}$$

Proof Since $\varphi(t)$ is a polynomial of degree not exceeding $-1 + \sum_{i=1}^{n}|S_i|$, (54) holds with f replaced by φ. That is,

$$a_{IJ}\varphi^{(J)}(\tau_I) + \sum_{i=1}^{n}\sum_{j\in S_i} a_{ij}\varphi^{(j)}(\tau_i) = 0. \tag{56}$$

Because of the interpolational property of φ, $\varphi^{(j)}(\tau_i) = f^{(j)}(\tau_i)$ for all $j \in S_i$, $i \in \{1, 2, \ldots, n\}$. Hence,

$$a_{IJ}\varphi^{(J)}(\tau_I) + \sum_{i=1}^{n}\sum_{j\in S_i} a_{ij}f^{(j)}(\tau_i) = 0. \tag{57}$$

Subtract (57) from (54), and the result follows. $\square$

Filling in missing data (Rob's original draft)

If $f(t)$ is required at many points, it may make sense to 'fill in' the missing data and thereby convert the Birkhoff problem to a pure hermite problem, for which the generalized barycentric formulae provide explicit answers. this turns out to be simple to do, as follows.

If there is only one missing piece, say $f^{(J)}(\tau_I)/J!$ is not known, then we proceed as before,

$$\frac{1}{2\pi i}\oint_c \sum_{i=1}^{n}\sum_{j=0}^{s_i-1}\gamma_{ij}\frac{f(z)}{(z-\tau_i)^{j+1}}dz = 0\,, \tag{58}$$

for all polynomials $f(z)$ of degree $\leqslant p = -2+\sum_{i=1}^{n} s_i$. Thus

$$\gamma_{IJ}\frac{f^{(J)}(\tau_I)}{J!} + \sum_{i=1}^{n}\sum_{\substack{j=0\\(i,j)\neq(I,J)}}^{s_i-1} a_{ij}f^{(j)}(\tau_i) = 0\,, \tag{59}$$

(where $a_{ij} = \gamma_{ij}/j!$ as before) giving, if $\gamma_{IJ}\neq 0$, a unique value for

$$f^{(J)}(\tau_I) = \frac{-1}{a_{IJ}}\sum_{i=1}^{n}\sum_{\substack{j=0\\(i,j)\neq(I,J)}}^{s_i-1} a_{ij}\,.f^{(j)}(\tau_i) \tag{60}$$

If $\gamma_{IJ} = 0$, then the problem is not poised (see section **??**).

If there are two missing pieces of data, say $f^{(J)}(\tau_I)$ and $f^{(L)}(\tau_K)$, then we use, once for (I,J) and again for (K,L), a $B(z)$ factor of degree 1 to suppress the residue at the other missing piece:

$$\frac{1}{2\pi i}\oint_c \frac{(z-b_0)f(z)}{w(z)}dz = 0\,, \tag{61}$$

for all polynomials of degree $\leqslant p = -3+\sum_{i=1}^{n} s_i$. We have therefore,

$$\sum_{i=1}^{n}\sum_{j=0}^{s_i-1} a_{ij}(b_0)f^{(j)}(\tau_i) = 0\,, \tag{62}$$

and we may hope to choose b_0 in order to set, first, $a_{IJ}(b_0^*) = 0$ whence

$$f^{(L)}(\tau_k) = \frac{-1}{a_{KL}}\sum_{i=1}^{n}\sum_{\substack{j=0\\(i,j)\neq(I,J)\\(i,j)\neq(K,L)}}^{s_i-1} a_{ij}(b_0^*)f^{(j)}(\tau_i)\,, \tag{63}$$

and then $b_0 = \widehat{b}_0$ making $a_{KL}(\widehat{b}_0) = 0$, giving a similar formula for $f^{(J)}(\tau_I)$.

Now that the idea is clear, we lay out the general case, where we have m missing pieces of data in sets $j \in S_i^c$, $1 \leqslant i \leqslant n$, as before. For each missing piece (I,J) we form

$$B^{IJ}(z) = b_0^{IJ} + b_1^{IJ}z + \ldots + b_{m-2}^{IJ}z^{m-1} + z^m\,, \tag{64}$$

and choose the $m-1$ free coefficients to suppress the residues at each of the other $m-1$ missing data locations, leading to

$$f^{(J)}(\tau_I)\frac{-1}{a_{IJ}^{IJ}}\sum_{i=1}^{n}\sum_{j\in S_i} a_{ij}^{IJ}f^{(j)}(\tau_i)\,. \tag{65}$$

Since the b_k^{IJ} appears linearly in the residues, the equations for choosing the b_k^{IJ} will also be linear. Suppose the partial fraction decomposition of

$$\frac{B^{IJ}(z)}{w(z)} = \sum_{i=1}^{n} \sum_{j=0}^{s_i-1} \frac{\widehat{\gamma}_{ij}}{(z-\tau_i)^{j+1}} \,, \tag{66}$$

then as before $a_{ij}^{IJ} = \dfrac{\widehat{\gamma}^{ij}}{j!}$. So, we need only compute the partial fraction decomposition.

We do this by local Taylor series:

$$B(z) = B(\tau_i) + B'(\tau_i)(z-\tau_i) + \ldots + \frac{B^{(s_i-1)}}{(s_i-1)!}(\tau_i)(z-\tau_i)^{s_i-1} + \ldots \tag{67}$$

whence

$$\begin{aligned}
\frac{B(z)}{w(z)} &= \sum_{i=1}^{n} \sum_{j=0}^{s_i-1} \sum_{k=0}^{j} \gamma_{ij}\beta_{ik}(z-\tau_i)^{k-j-1} \\
&= \sum_{i=1}^{n} \sum_{k=0}^{s_i-1} \sum_{j=k}^{s_i-1} \gamma_{ij}\beta_{ik}(z-\tau_i)^{k-j-1} \\
&\qquad\qquad \begin{matrix}\mu=j-k\\ j=\mu+k\end{matrix} \\
&= \sum_{i=1}^{n} \sum_{k=0}^{s_i-1} \sum_{\mu=0}^{s_i-1-k} \gamma_{i,\mu+k}\beta_{ik}(z-\tau_i)^{-\mu-1} \\
&= \sum_{i=1}^{n} \sum_{\mu=0}^{s_i-1} \left(\sum_{k=0}^{s_i-1-\mu} \gamma_{i,\mu+k}\beta_{ik} \right)(z-\tau_i)^{-\mu-1}
\end{aligned} \tag{68}$$

$$\therefore \quad \widehat{\gamma}_{ij} = \sum_{k=0}^{s_i-1-j} \gamma_{i,\mu+k}\beta_{ik} \tag{69}$$

and

$$\beta_{ik} = \frac{B^{(k)}(\tau_i)}{k!} = \sum_{j=k}^{m} \binom{m}{j} \tau_i^{j-k}.\, b_j \tag{70}$$

$$\therefore \quad \widehat{\gamma}_{ij} = \sum_{k=0}^{s_i-1-j} \sum_{k=0}^{m} \binom{m}{j} \gamma_{i,\mu+k}\tau_i^{j-k} b_j \,. \tag{71}$$

4.3 Numerical quadrature and differentiation

4.4 Root-finding

Write the first draft $\surd$

We may be required to compute the roots of a univariate polynomial, specified by its values in the Hermite-Birkhoff interpolation basis. There are strong numerical and structural advantages in working with the given data directly without completing the interpolation and converting to the familiar monomial basis [](references; Higham, Rob's papers and more...). Numerically stable sets of basis-preserving root-finding algorithms, in the Hermite and Lagrange cases, have been presented in [] (azar's thesis, our paper;Geometric Application...).

If the data is given in the general Hermite-Birkhoff case, we can use the approach of Section 4.2 to first fill in the missing data. Having done that, without completing the interpolation, we have converted the original problem to the Hermite case. Then, we can use the existing algorithms to compute all the simple, isolated roots of the given polynomial by solving a generalized eigenvalue problem. This is done by constructing a pair of regular matrices, known as the companion pencil matrices. The finite generalized eigenvalues of this matrix pencil are the roots of the original polynomial.

Example 2

4.5 Rational interpolation

The approach described in this paper can be generalized to the rational polynomial case. Henceforth, we assume $R(z) = f(z)/q(z)$ is the rational polynomial with $q(z)$ fixed and not identically zero.

Lemma 3 *Let $w(z)$ be defined as in (7). Suppose $\deg q(z) \leqslant -1 + \sum_{i=1}^{n} s_i$. Put $\sigma_{ij} = \frac{q^{(j)}(\tau_i)}{j!}$ for brevity. Then*

$$\frac{q(z)}{w(z)} = \sum_{i=1}^{n} \sum_{j=0}^{s_i-1} \frac{\alpha_{ij}}{(z-\tau_i)^{j+1}} \tag{72}$$

where

$$\alpha_{ij} = \sum_{k=0}^{s_i-1-j} \gamma_{i,j+k}\sigma_{ik} \,. \tag{73}$$

Moreover, if $q(\tau_i) \neq 0$, and γ_{i,s_i-1} is as defined in (19) then

$$\alpha_{i,s_i-1} = \gamma_{i,s_i-1}\sigma_{i,0} \neq 0. \tag{74}$$

Proof By the Hermite interpolation formula,

$$\frac{q(z)}{w(z)} = \sum_{i=1}^{n} \sum_{j=0}^{s_i-1} \sum_{k=0}^{j} \gamma_{ij}\sigma_{ik}(z-\tau_i)^{k-j-1} \tag{75}$$

Interchanging the order of the second and third sums,

$$\frac{q(z)}{w(z)} = \sum_{i=1}^{n} \sum_{k=0}^{s_i-1} \sum_{j=k}^{s_i-1} \gamma_{ij}\sigma_{ik}(z-\tau_i)^{k-j-1} \,. \tag{76}$$

Putting $j' = j - k$,

$$\frac{q(z)}{w(z)} = \sum_{i=1}^{n} \sum_{k=0}^{s_i-1} \sum_{j'}^{s_i-1-k} \gamma_{i,j'+k}\sigma_{ik}(z-\tau_i)^{-j'-1} \tag{77}$$

Interchanging the second and third sums again and dropping the $'$,

$$\frac{q(z)}{w(z)} = \sum_{i=1}^{n} \sum_{j=0}^{s_i-1} \left(\sum_{k=0}^{si-1-j} \gamma_{i,j+k}\sigma_{ik} \right) (z-\tau_i)^{-j-1} \tag{78}$$

as claimed.

Now,

$$\alpha_{i,s_i-1} = \sum_{k=0}^{0} \gamma_{i,s_i-1+k}\sigma_{i,k} = \gamma_{i,s_i-1}\sigma_{i,0} \neq 0\,, \tag{79}$$

also as claimed, since $\sigma_{i,0} = q(\tau_i) \neq 0$ □

Lemma 4 *If* $\deg q(z) \leqslant -2 - m + \sum_{i=1}^{n} s_i$, *then for all polynomials* $B(z)$ *of degree* $\leqslant m$ *and all polynomials* $f(z)$ *of degree* $\leqslant -2-m+\sum_{i=1}^{n} s_i$, *if the contour* C *is taken large enough to enclose all* τ_i

$$0 = \frac{1}{2\pi i}\oint_C \frac{B(z)f(z)}{w(z)}dz = \frac{1}{2\pi i}\oint_C \frac{B(z)q(z)}{w(z)} \cdot \frac{f(z)}{q(z)}dz\,. \tag{80}$$

Proof The proof follows by degree counting and Lemma 1. □

The idea of Theorem 2 can be generalized to the rational polynomial case.

Proposition 1 *Let, as before,* $R(z) = f(z)/q(z)$ *be the rational polynomial with* $q(z)$ *fixed. If* $B(z)$ *can be chosen as* $B^{IJ}(z) = \sum_{k=0}^{m-1} b_k^{IJ} z^k$ *so that*

$$\frac{B^{IJ}(z)q(z)}{w(z)} = \frac{-1}{(z-\tau_I)^{J+1}} + \sum_{i=1}^{n}\sum_{j\in S_i} \frac{\alpha_{ij}^{IJ}}{(z-\tau_i)^{j+1}}\,, \tag{81}$$

Fix the notation, we have avoided using the notation S_I^c *before the following* $\surd$

where $J \notin S_I$, *and as before* m *is the number of missing data items, then*

$$\frac{R^{(J)}(\tau_I)}{J!} = \sum_{i=1}^{n}\sum_{j\in S_i} \alpha_{ij}^{IJ}\rho_{ij} \tag{82}$$

for every polynomial $f(z)$ *of degree* $\leqslant -1-m+\sum_{i=1}^{n} S_i$ *and with*

$$\frac{R^{(j)}(\tau_i)}{j!} = \rho_{ij} \tag{83}$$

given for $1 \leqslant i \leqslant n$, $j \in S_i$. *Recall also that for an arbitrary* ι, $R^{(\iota)}(\tau_i) = \frac{d^\iota}{dz^\iota}\left(\frac{f(z)}{q(z)}\right)\Big|_{\{z=\tau_i\}}$.

Proof Replace m with $m-1$ in Lemma 4 (note that the residues when $q(z) = 0$ are zero), and apply Cauchy's Theorem, it follows that

$$\begin{aligned} 0 =& \frac{1}{2\pi i}\oint \frac{B(z)q(z)}{w(z)} \cdot \frac{f(z)}{q(z)}dz \\ =& \frac{1}{2\pi i}\oint \left(\frac{-1}{(z-\tau_I)^{(J+1)}} + \sum_{i=1}^{n}\sum_{j\in S_i} \frac{\alpha_{ij}^{IJ}}{(z-\tau_i)^{j+1}} \right) \frac{f(z)}{q(z)}dz \\ =& \frac{-R^{(J)}(\tau_I)}{J!} + \sum_{i=1}^{n}\sum_{j\in S_i} \alpha_{ij}^{IJ}\rho_{ij}\,. \end{aligned} \tag{84}$$

□

5 Conditioning, well-posedness and poisedness

6 Concluding remarks

(connection to divided differences)

7 Bibliography

References

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