Calculus 1301b

Note Title 28/01/2011

- Prof Shemyakova is not available today
- My name is Rob Corless
- today's lecture notes will be made available on www.apmaths.uwo.ca/~rcorless under "Teaching"

Today: more section 9.3 - separable eq'ns.

Mixing problems (eg Example 6, p 585
problems 41–44 p 587).

blah blah blah ← this is the hard bit (interesting bit)

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

$$\frac{1}{150-y}\frac{dy}{dt} = \frac{1}{200}$$

$$\int \frac{1}{150-y}\frac{dy}{dt}\,dt = \int \frac{1}{200}\,dt = \frac{1}{200}t + c$$

change of variables

$u = y(t)$ $\quad du = \frac{dy}{dt}\,dt$

$$\int \frac{1}{150-u}\,du = -\ln(150-u) + C_1$$

(or $-\ln|150-u|$

but watch out if $u = 150$)

$$-\ln(150-u) = \frac{1}{200}\cdot t + \underbrace{C - C}_{K}$$

$$-\ln(150-y(t)) = $$

$$= \frac{1}{200} t + K$$

1^{st} order eq'n $\Rightarrow$ one constant of integration

$$\ln(150-y(t)) = -\frac{1}{200} t + K$$

$$e^{\ln(150-y(t))} = e^{-\frac{1}{200}t + K}$$

or $150 - y(t) = Ce^{-\frac{t}{200}}$ $\left(C = e^{k} \text{ here}\right)$.

or $150 - Ce^{-t/200} = y(t)$

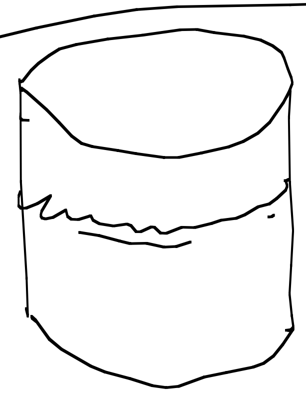
blah: "Let $y(t)$ be the amount of salt in kg at time t"

$y(t) = 150 - Ce^{-t/200}$

$$\boxed{\lim_{t \to \infty} y(t) = 150}$$

Problem set-up in more detail (modelling)

A tank



contains 20 kg of salt dissolved in 5000L of water

$$y(t) = 150 - C \cdot e^{t/200}$$

at $t=0$ $y(t) = 20$

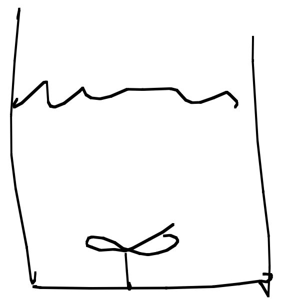
$$20 = 150 - C \cdot 1 \quad \therefore \quad C = 150 - 20 = 130$$

$$\boxed{\therefore \quad y(t) = 150 - 130e^{-t/200}}$$

Why does $\lim_{t \to \infty} 150 - 130\underbrace{e^{-t/200}}_{\text{goes to } 0.} = 150$?

" Brine that contains 0.03 kg/L of salt enters the tank at a rate of 25 L/min. "

"The solution is kept thoroughly mixed

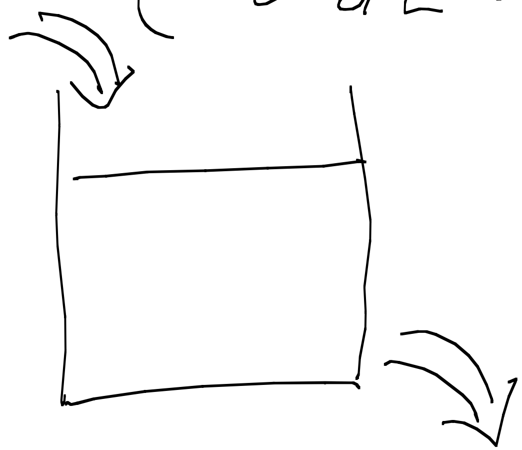


ideal mixer, instantly brings brine to evenly distributed.

and drains from the tank at the same rate (25 L/m)."

How much salt remains in the tank after 1 hour (60 min).

(salt in) $(0.03\ kg/L \times 25\ L)/min$



$\left(y(t)/5000\ L\right) \times 25\ L/min$ (salt out)

$\|$

concentration $= \dfrac{\text{amount of salt}}{\text{total volume}}$

$$\boxed{\text{rate of change of salt} = \text{salt in per minute} - \text{salt out per minute}}$$

$$\frac{dy}{dt} = (0.03)(25) - y(t) \cdot \frac{\times 25}{5000}$$

$$= 0.75 - \frac{y(t)}{200}$$

AHA!

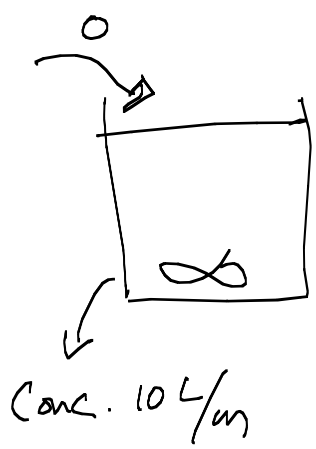
A similar problem : #41 in text

A tank contains 1000 L of brine, 15 kg salt initially dissolved.

Pure water enters @ 10 L/min

(drains @ same rate, thoroughly mixed)

concentration $\frac{y(t)}{1000}$ ($y(t)$ = amount of salt at t minutes)



rate of change of salt = amount in per minute − amount out per minute

$$\frac{dy}{dt} = 0 - \frac{y(t)}{1000} \cdot 10$$

$$= -\frac{y(t)}{100}$$

$$\frac{dy}{dt} = -\frac{1}{100}y \qquad \frac{1}{y}\frac{dy}{dt} = -\frac{1}{100}$$

key

$$\therefore \int \frac{1}{y}\frac{dy}{dt}\,dt = \int -\frac{1}{100}\,dt$$

$$\therefore \ln y = -\frac{1}{100}t + c$$

$$\therefore y = Ke^{-t/100}$$

$y(0) = 15\text{ kg}$ (initial amount of salt)

$\therefore\ 15\text{ kg} = K \cdot e^{-0/100} = K$

$\therefore\ y(t) = 15e^{-t/100}$

b) amount after 20 minutes?

$y(20) = 15e^{-20/100} = 15e^{-1/5}$

In general $\dfrac{dy}{dx} = f(y)\, g(x)$

$$\frac{1}{f(y)} \frac{dy}{dx} = g(x)$$

$$\therefore \int \frac{1}{f(y)} \frac{dy}{dx}\, dx = \int g(x)\, dx$$

$$= \int \frac{1}{f(u)}\, du$$

Change of variables : $u = g(x)$ $du = \frac{dy}{dx} dx$

$$\int 2x \sin(x^2)\, dx$$

$u = x^2$ $du = 2x\, dx$

$$\Rightarrow \int \sin(u)\, du$$