Symmetry

1. It is possible to define an order parameter.
2. It is possible to describe the system with a free energy.
3. The free energy must be consistent with the high temperature symmetry properties of the system.
4. The free energy must be analytic. In addition, the expansion coefficients must be regular functions of the temperature.

Close to T_c the free energy can be expanded in powers of the order parameter

free energy

order parameter. must be small

$$F(\Psi) = \sum_{n=0}^{\infty} a_{2n} \Psi^{2n}$$

expansion coefficients are phenomenological parameters that depend on T and microscopics

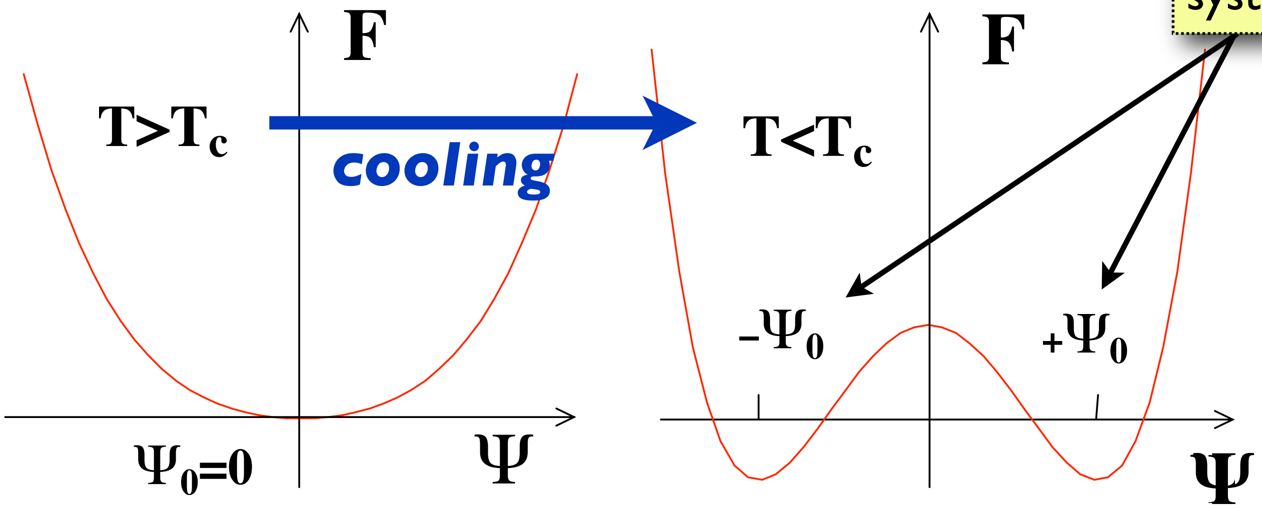
Order parameter must be small for the expansion to converge.

Symmetry

1. It is possible to define an order parameter.
2. It is possible to describe the system with a free energy.
3. The free energy must be consistent with the high temperature symmetry properties of the system.
4. The free energy must be analytic. In addition, the expansion coefficients must be regular functions of the temperature.

Continuous transition: No odd powers allowed in the free energy expansion.

Spontaneous symmetry breakdown: The ground state of the system has lower symmetry than the free energy shows.



equivalent states but the system can pick only one

Symmetry

1. It is possible to define an order parameter.
2. It is possible to describe the system with a free energy.
3. The free energy must be consistent with the high temperature symmetry properties of the system.
4. The free energy must be analytic. In addition, the expansion coefficients must be regular functions of the temperature.

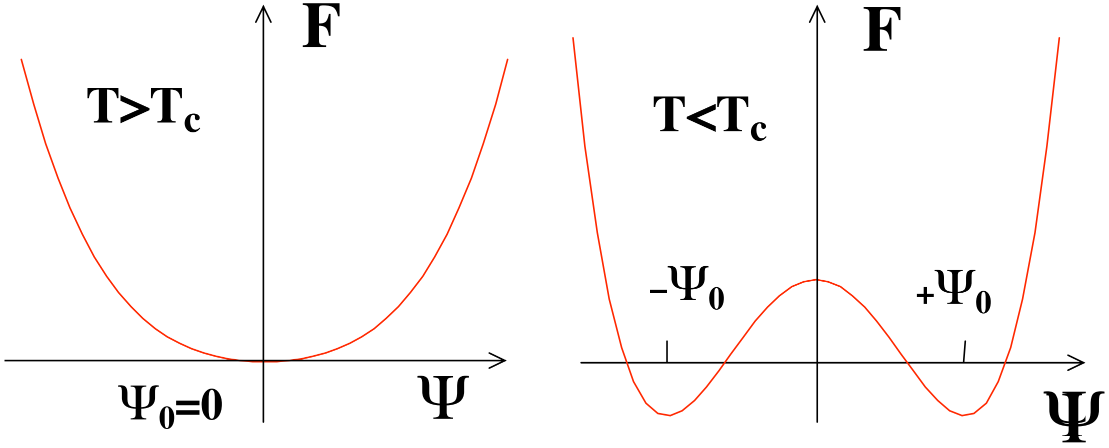
The second order term is dominant: $a_2(T)$ must vary smoothly from $a_2(T) < 0 \text{ for } T < T_c$ to $a_2(T) > 0 \text{ when } T > T_c$, with $a_2(T = T_c) = 0$. This implies that $a_2(T) \sim (T - T_c)$.

First minimize the free energy

$$\frac{\partial F}{\partial \Psi} = 2a_2\Psi + 4a_4\Psi^3 = 0$$

solutions are

$$\boxed{\Psi_0 = 0 \text{ and } \Psi_0 = \sqrt{-\frac{a_2}{2a_4}}}$$



Symmetry

1. It is possible to define an order parameter.
2. It is possible to describe the system with a free energy.
3. The free energy must be consistent with the high temperature symmetry properties of the system.
4. The free energy must be analytic. In addition, the expansion coefficients must be regular functions of the temperature.

$$\frac{\partial F}{\partial \Psi} = 2a_2\Psi + 4a_4\Psi^3 = 0$$

also shows that a_4 >0. Otherwise $\Psi \rightarrow \infty$ would minimize the free energy and the above would be useless.

Next, we Taylor expand the expansion coefficients around T_c

$$a_2(T) \approx a_2(T_c) + (T - T_c)\frac{\partial a(T)}{\partial T}\bigg|_{T=T_c} + \frac{1}{2!}(T - T_c)^2\frac{\partial^2 a(T)}{\partial T^2}\bigg|_{T=T_c} + \dots\,,$$

$$a_4(T) \approx a_4(T_c) + (T - T_c)\frac{\partial a_4(T)}{\partial T}\bigg|_{T=T_c} + \frac{1}{2!}(T - T_c)^2\frac{\partial^2 a_4(T)}{\partial T^2}\bigg|_{T=T_c} + \dots\,.$$

Symmetry

1. It is possible to define an order parameter.
2. It is possible to describe the system with a free energy.
3. The free energy must be consistent with the high temperature symmetry properties of the system.
4. The free energy must be analytic. In addition, the expansion coefficients must be regular functions of the temperature.

As argued above $a_4(T) > 0$, and we can approximate $a_4 = a_4(T_c) = \text{constant}$.

Note: The above is enough to guarantee the finiteness of the order parameter.

a_2 must change its sign upon cooling below T_c. Then we have

$$\left.\frac{\partial a_2}{\partial T}\right|_{T=T_c} = const. > 0 \quad \text{and} \quad a_2(T) \sim (T - T_c)$$

Landau theory

1. It is possible to define an order parameter.
2. It is possible to describe the system with a free energy.
3. The free energy must be consistent with the high temperature symmetry properties of the system. Mathematically speaking, the Hamiltonian must commute with the symmetry group of the high temperature phase.
4. The free energy must be analytic. In addition, the expansion coefficients must be regular functions of the temperature.

We could now simply compute the critical exponents using F. They turn out to be the mean field exponents (van der Waals). Homework.

$$\psi \sim (T_c - T)^{\beta}$$

$$\psi \sim H^{1/\delta}$$

$$\chi \sim (T_c - T)^{-\gamma}$$

$$c_v \sim (T_c - T)^{-\alpha}$$

Landau theory is mean field

Important:

The Landau theory is a mean-field theory since it does not take into account ***spatial inhomogeneities*** or ***thermal fluctuations***.

To take account of ***inhomogeneities***, we have to let the order parameter become space dependent, i.e.,

$$\Psi = \Psi(\vec{x})$$

Conceptually, we should think of Ψ as a ***coarse-grained*** order parameter, i.e., it is defined only over a certain length scale. Example: block spins in an Ising model. Block no larger than characteristic length.

We must define a short wavelength (ultraviolet) cut-off for Ψ in such a way that it varies smoothly in space; the order parameter cannot fluctuate on smaller length scales than the cut-off.

In practice, the cut-off may often be thought of as the lattice spacing.

Landau theory: Spatial inhomogeneties

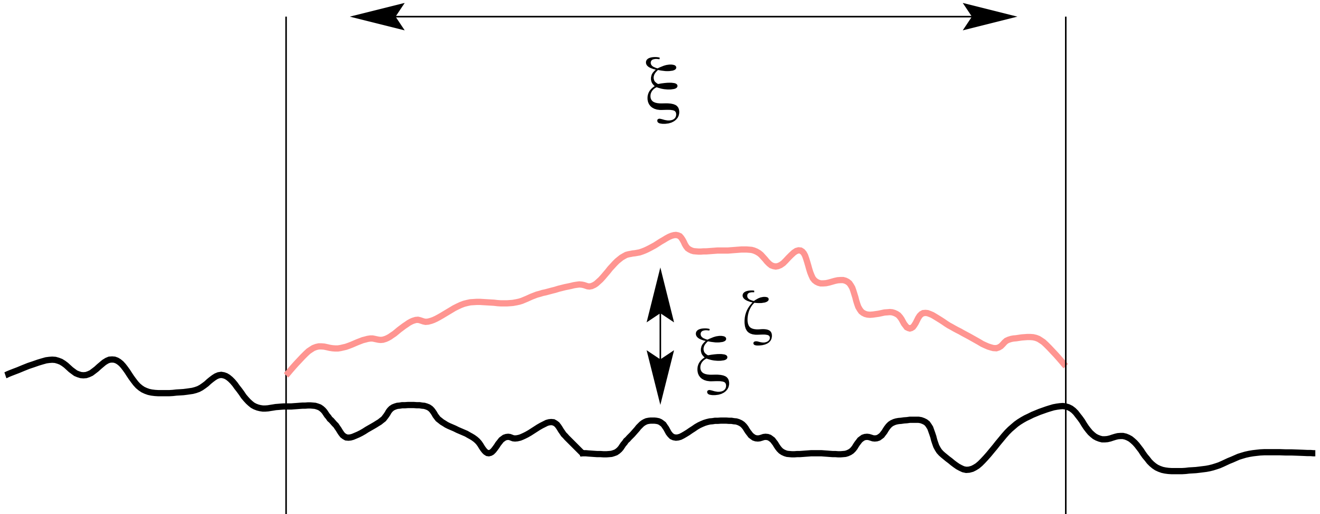
The response of the system to spatial inhomogeneities is **internal rigidity**.

Physically:

Large differences between neighboring points are unfavorable. In other words, rigidity is a generic property of the system, and therefore there must an energy cost associated with spatial inhomogeneities. That is why we defined

$$\Psi = \Psi(\vec{x})$$

Example:



Landau theory: Free energy functional

Important:

Since the order parameter is a smooth and slowly varying function, we can take the **rigidity** of the system into account by making a gradient expansion and retaining only the lowest order term compatible with the symmetry properties.

The validity of the truncation of the gradient expansion depends on the smoothness and slowness of the order parameter variations.

Since the order parameter is now a local variable, the free energy becomes a functional

$$\boxed{F[\Psi] = \int d^d\vec{x} \left[F(\Psi) + \frac{1}{2} K (\nabla\Psi)^2 \right]}$$

K is a phenomenological parameter describing the rigidity of the system.
It must be positive for the free energy to be bounded from below