4 a) The second translation theorem states

$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$

In this case, $f(t-a) = t-1$, with $a=1$

$\Rightarrow f(t-1) = t-1 \Rightarrow f(t) = t$

$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2}$

$\Rightarrow e^{-s}\cdot\frac{1}{s^2}$ is the answer

b) This time, $a = \pi$

$\Rightarrow f(t-\pi) = \cos(2t) = \cos(2(t-\pi)+2\pi)$

$= \cos(2(t-\pi))$

$\Rightarrow f(t) = \cos 2t$

$F(s) = \frac{s}{s^2+4}$

$\Rightarrow \mathcal{L}\{\cos(2t)H(t-\pi)\} = \frac{s}{s^2+4}e^{-\pi s}$

c) ~~The first translation theorem states~~

$\cancel{\mathcal{L}\{e^{at}f(t)\} = F(s-a)}$

~~In this case, $a=-2$~~

Apologies, Too lazy to start a new page.

We reverse the second translation theorem. In this case, $a=2$,

$F(s) = \frac{1}{s^2}$

$\Rightarrow f(t) = t \Rightarrow f(t-2) = t-2$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \cancel{\mathcal{L}^{-1}}\mathcal{L}[[f(t-a)H(t-a)]]$

$\mathcal{L}^{-1}\{e^{-2s}F(s)\} = (t-2)H(t-2)$

d) $a = \pi, \quad F(s) = \frac{1}{s^2+1}$

$\Rightarrow f(t) = \sin t, \quad f(t-\pi) = \sin(t-\pi)$

$\mathcal{L}^{-1}\{\frac{e^{-\pi s}}{s^2+1}\} = f(t-\pi)H(t-\pi) = \sin(t-\pi)H(t-\pi)$

5. $\mathcal{L}\{y''+4y\} = s^2\mathcal{L}\{y\} - y'(0) - sy(0) + 4\mathcal{L}\{y\}$

$= (s^2+4)\mathcal{L}\{y\} - s$

$\mathcal{L}\{\sin t\, H(t-2\pi)\}$: $f(t-2\pi) = \sin t = \sin((t-2\pi)+2\pi)$

$= \sin(t-2\pi)$

$\Rightarrow f(t) = \sin t, \quad F(s) = \frac{1}{s^2+1}$

$\Rightarrow \mathcal{L}\{\sin t\, H(t-2\pi)\} = e^{-2\pi s}\frac{1}{s^2+1}$

Above two equal, so

$(s^2+4)\mathcal{L}\{y\} - s = \frac{1}{s^2+1}e^{-2\pi s}$

$\Rightarrow \mathcal{L}\{y\} = \frac{s}{s^2+4} + \frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+4)}\cdot e^{-2\pi s}$

Use partial fractions to say that

$\frac{1}{s^2+1}\cdot\frac{1}{s^2+4} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)$

$\mathcal{L}\{y\} = \frac{s}{s^2+4} + \frac{1}{3}\frac{1}{(s^2+1)}e^{-2\pi s} - \frac{1}{3}\frac{1}{(s^2+4)}e^{-2\pi s}$

$\mathcal{L}^{-1}\{\frac{s}{s^2+4}\} = \cos 2t$

$\mathcal{L}^{-1}\{\frac{1}{3}\frac{1}{s^2+1}e^{-2\pi s}\} = \frac{1}{3}\sin(t-2\pi)H(t-2\pi)$

$= \frac{1}{3}\sin t\, H(t-2\pi)$

$\mathcal{L}^{-1}\{\frac{1}{3}\frac{1}{s^2+4}e^{-2\pi s}\} = \frac{1}{3}\sin(2(t-2\pi))H(t-2\pi)$

$= \frac{1}{3}\sin 2t\, H(t-2\pi)$

$\Rightarrow \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} = y = \cos 2t + (\frac{1}{3}\sin t - \frac{1}{3}\sin 2t)H(t-2\pi)$

6. $\mathcal{L}\{EI y^{(4)}\} = EI(s^4 Y - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0))$

where $Y = \mathcal{L}\{y\}$ to save excessive use of pen ink

$= EI(s^4 Y - s y''(0) - y'''(0))$

$\mathcal{L}\{w_0 \delta(x - \frac{L}{2})\} = w_0 e^{-\frac{L}{2}s}$

Two transforms above should equal. But let us have $c_1 = y''(0)$, $c_2 = y'''(0)$ for convenience

$$EI(s^4 Y - s c_1 - c_2) = w_0 e^{-\frac{L}{2}s}$$
$$\Rightarrow s^4 Y - s c_1 - c_2 = \frac{w_0}{EI} e^{-\frac{L}{2}s}$$
$$\Rightarrow Y = \frac{1}{s^4}\frac{w_0}{EI} e^{-\frac{L}{2}s} + \frac{c_1}{s^3} + \frac{c_2}{s^4}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^4}\frac{w_0}{EI} e^{-\frac{L}{2}s}\right\} + \mathcal{L}^{-1}\left\{\frac{c_1}{s^3}\right\} + \mathcal{L}^{-1}\left\{\frac{c_2}{s^4}\right\}$$

$$y = \frac{w_0}{6EI}(t - \tfrac{L}{2})^3 H(t - \tfrac{L}{2}) + \frac{c_1 t^2}{2} + \tfrac{1}{6} c_2 t^3$$

Everything should be in x, not t. My bad

$$y = \frac{w_0}{6EI}(x - \tfrac{L}{2})^3 H(x - \tfrac{L}{2}) + \tfrac{1}{2} c_1 x^2 + \tfrac{1}{6} c_2 x^3$$

$$y'' = \frac{w_0}{EI}(x - \tfrac{L}{2}) H(x - \tfrac{L}{2}) + \frac{w_0}{EI}(x - \tfrac{L}{2})^2 \delta(x - \tfrac{L}{2}) + \frac{w_0}{6EI}(x - \tfrac{L}{2})^3 \delta'(x - \tfrac{L}{2}) + c_1 + c_2 x$$

Note: $H'(x - \frac{L}{2}) = \delta(x - \frac{L}{2})$, $\delta'(x - \frac{L}{2}) = -\frac{1}{(x - \frac{L}{2})}\delta(x - \frac{L}{2})$

$$y''(L) = \frac{L w_0}{2EI} + c_1 + c_2 L = 0 \qquad (1)$$

since $\delta(L - \frac{L}{2}) = \delta(\frac{L}{2}) = 0$

$$y'''(x) = \frac{w_0}{EI} H\left(x - \frac{L}{2}\right) + \text{junk} + c_2$$

Explanation of 'junk': As we saw earlier, every derivative of $H(x-\frac{L}{2})$ ~~taken at~~ yields some multiple of $\delta(x-\frac{L}{2})$. We evaluate these at L, which means that $\delta(L-\frac{L}{2})=0$ and 'junk' will always be 0. when evaluated at L.

$$\Rightarrow y'''(L) = \frac{w_0}{EI} + c_2 = 0 \Rightarrow c_2 = -\frac{w_0}{EI}$$

Going back to (1),

$$\frac{Lw_0}{2EI} + c_1 - \frac{Lw_0}{EI} = 0 \Rightarrow c_1 = \frac{Lw_0}{2EI}$$

Now that we know c_1, c_2, we know that y is

$$y = \frac{w_0}{EI}\left[\frac{1}{6}\left(x-\frac{L}{2}\right)^3 H\left(x-\frac{L}{2}\right) + \frac{L}{4}x^2 - \frac{1}{6}x^3\right.$$