Problem 4

$$f(t)+\int_0^t (t-\tau)f(\tau)\,d\tau = 1$$

$$\mathcal{L}\{f(t)\}+\mathcal{L}\{\underbrace{\textstyle\int_0^t (t-\tau)f(\tau)\,d\tau}_{=(g*f)(t)}\}=\mathcal{L}\{1\}$$

where $g(t)=t$

$$\mathcal{L}\{(g*f)(t)\} = \mathcal{L}\{g\}\cdot\mathcal{L}\{f\}$$
$$= \mathcal{L}\{t\}\cdot\mathcal{L}\{f(t)\}$$
$$= \frac{1}{s^2}\mathcal{L}\{f(t)\}$$

So, $\left(1+\frac{1}{s^2}\right)\mathcal{L}\{f(t)\}=\frac{1}{s}$

$$\mathcal{L}\{f(t)\} = \frac{1}{s(1+1/s^2)} = \frac{1}{s+1/s} = \frac{s}{s^2+1}$$

$$\therefore f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$
$$= \underline{\underline{\cos t}}$$

Problem 5

$$y'(t) = 1-\sin t - \int_0^t y(\tau)\,d\tau \quad , y(0)=0$$

$$\mathcal{L}\{y'(t)\} = \mathcal{L}\{1\}-\mathcal{L}\{\sin t\}-\mathcal{L}\left\{\int_0^t y(\tau)\,d\tau\right\}$$

$$\mathcal{L}\left\{ \int_0^t y(\tau)\,d\tau \right\} = \mathcal{L}\{(f*y)(t)\} \quad \text{where } f(t)=1$$

$$= \mathcal{L}\{f\}\cdot\mathcal{L}\{y\}$$

$$= \mathcal{L}\{1\}\cdot\mathcal{L}\{y(t)\}$$

$$= \frac{1}{s}\mathcal{L}\{y(t)\}$$

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - \cancel{y(0)}^{\,0}$$

So, $(s+1/s)\,\mathcal{L}\{y(t)\} = \frac{1}{s} - \frac{1}{s^2+1}$ $\qquad s + 1/s = \frac{1}{s}(s^2+1)$

$$\mathcal{L}\{y(t)\} = \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}$$

$$\therefore\ y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$$

$$\mathcal{L}^{-1}\left\{(-1)^n\frac{d^n}{ds^n}F(s)\right\} = t^n f(t)$$

So,

$$y(t) = \underline{\underline{\sin t - \tfrac{1}{2}t\sin t}}$$

Problem 6

a) $(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau \quad , t \geq 0$

let $z = t-\tau \Rightarrow dz = d\tau$

$= \int_0^t f(z)g(t-z)dz$

$= \int_0^t g(t-z)f(z)dz$

$= (g*f)(t)$

b) $f*(g+h) = \int_0^t f(t-\tau)[g(\tau)+h(\tau)]d\tau$

$= \int_0^t f(t-\tau)g(\tau)d\tau + \int_0^t f(t-\tau)h(\tau)d\tau$

$= f*g + f*h$