1. For these questions, we use theorem 4.8 in the book

a) $f(t) = \cos 2t \qquad \mathcal{L}\{f(t)\} = F(s) = \cancel{\cos}\ \frac{s}{s^2+4}$

$$\mathcal{L}\{t\,f(t)\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = (-1)\left[\frac{1}{s^2+4} - \frac{2s^2}{(s^2+4)^2}\right]$$

$$=(-1)\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right] = \frac{s^2-4}{(s^2+4)^2}$$

b) $f(t) = \sinh t \qquad F(s) = \frac{1}{s^2-1}$

$$\mathcal{L}\{t^2 f(t)\} = \frac{d^2}{ds^2}\left(\frac{1}{s^2-1}\right) = \frac{d}{ds}\left(\frac{-2s}{(s^2-1)^2}\right)$$

$$= \left[\frac{-2}{(s^2-1)^2} + \frac{8s^2}{(s^2-1)^3}\right] = \frac{-2s^2+2+8s^2}{(s^2-1)^3} = \frac{6s^2+2}{(s^2-1)^3}$$

c) $f(t) = e^{2t}\sin(6t)$

Use first translation theorem

let $g(t) = \sin(6t) \Rightarrow G(s) = \frac{6}{s^2+36}$

$$F(s) = G(s-2) = \frac{6}{(s-2)^2+36} = \frac{6}{s^2-4s+40}$$

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}\left(\frac{6}{s^2-4s+40}\right) = \frac{6(2s-4)}{(s^2-4s+40)^2}$$

2. For these questions, we use the convolution theorem. I will group every term in terms of τ as $f(t)$ and everything in terms of $t-\tau$ as $g(t)$

a) $f(t) = e^t, \quad g(t) = 1$

$F(s) = \frac{1}{s-1} \qquad G(s) = \frac{1}{s}$

$\mathcal{L}\{f*g\} = F(s)\,G(s) = \frac{1}{s} \cdot \frac{1}{s-1}$

b) $f(t) = e^{-t}\cos t, \quad g(t) = 1$

$F(s) = \frac{(s+1)}{(s+1)^2+1}, \qquad G(s) = \frac{1}{s}$

$\mathcal{L}\{f*g\} = \frac{1}{s} \cdot \frac{s+1}{(s+1)^2+1}$

c) $f(t) = t, \qquad g(t) = e^t$

$F(s) = \frac{1}{s^2} \qquad G(s) = \frac{1}{s-1}$

$\mathcal{L}\{f*g\} = \frac{1}{s^2(s-1)}$

3. ~~We find $\int \ln x\,dx$ and use it.~~

~~Integrate by parts~~

~~$u = \ln x, \quad dv = 1$~~

~~$du = \frac{1}{x}, \quad v = x$~~

~~$\int \ln x\,dx = x\ln x - \int 1\,dx = x\ln x - x + C$~~

~~$\ln\frac{s-3}{s+1} = \ln(s-3) - \ln(s+1) = \frac{d}{ds}F(s)$~~

~~$\Rightarrow \int \ln(s-3) - \ln(s+1)\,ds = F(s)$~~

~~$= (s-3)\ln(s-3) - (s-3) - (s+1)\ln(s+1)$~~

~~$+ (s+1) + C$~~

3. $\mathcal{L}^{-1}\{\ln(\frac{s-3}{s+1})\}$ let $F(s) = \ln(\frac{s-3}{s+1})$

Then $\mathcal{L}^{-1}\{\frac{d}{ds} F(s)\} = -t f(t)$

$= \mathcal{L}^{-1}\{\frac{1}{s-3} - \frac{1}{s+1}\} = e^{3t} - e^{-t}$

$\Rightarrow f(t) = \frac{1}{t}(e^{-t} - e^{3t})$