Problem 1

a) $$\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$= \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{A(x^2-5x+6) + B(x^2-4x+3) + C(x^2-3x+2)}{(x-1)(x-2)(x-3)}$$

Grouping terms of the same order

x^2: $A + B + C = 0$

x^1: $-5A - 4B - 3C = 0$

x^0: $6A + 3B + 2C = 1$

Solving this system of equations gives,

$A = \frac{1}{2}$, $B = -1$, $C = \frac{1}{2}$

$$\therefore \frac{1}{(x-1)(x-2)(x-3)} = \frac{1/2}{(x-1)} + \frac{-1}{(x-2)} + \frac{1/2}{(x-3)}$$

b) $$\frac{1}{x^2-2x^2-x} = \frac{-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1) + Bx}{x(x+1)}$$

x^1: $A + B = 0$

x^0: $A = -1 \Rightarrow B = 1$

$$\therefore \frac{1}{x^2-2x-x} = \frac{-1}{x} + \frac{1}{x+1}$$

Problem 2

a) $f(s) = \dfrac{3}{s^2+4}$

The idea is to put the function into a general form found in a transform table

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} = \frac{3}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \underline{\frac{3}{2}\sin 2t}$$

b) $f(s) = \dfrac{2}{s^2+3s-4} = \dfrac{2}{(s+4)(s-1)} = \dfrac{A}{(s+4)} + \dfrac{B}{(s-1)}$

$$= \frac{A(s-1)+B(s+4)}{(s+4)(s-1)}$$

s^1: $A+B=0$

s^0: $-A+4B=2$

$\Rightarrow A = -2/5,\ B = 2/5$

$$\therefore f(s) = \frac{-2/5}{(s+4)} + \frac{2/5}{(s-1)}$$

$$\mathcal{L}^{-1}\{f(s)\} = -2/5\,\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + 2/5\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \underline{-2/5\, e^{-4t} + 2/5\, e^{t}}$$

c) $f(s) = \dfrac{2s+2}{s^2+2s+5}$

completing the square

$(s^2+2s+1)+5-1 = (s+1)^2+4$

$$= \frac{2(s+1)}{(s+1)^2+4}$$

$$\mathcal{L}^{-1}\{f(s)\} = 2\,\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} = \underline{2e^{-t}\cos 2t}$$

d) $f(s) = \dfrac{1-2s}{s^2+4s+5}$

again completing the square,

$(s^2+4s+4)+5-4$

$= (s+2)^2+1$

$= \dfrac{1-2s}{(s+2)^2+1}$

$= \dfrac{5}{(s+2)^2+1} - \dfrac{2(s+2)}{(s+2)^2+1}$ since $1-2s = 5-2(s+2)$

$\mathcal{L}^{-1}\{f(s)\} = 5\mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\dfrac{(s+2)}{(s+2)^2+1}\right\}$

$= 5e^{-2t}\sin t - 2e^{-2t}\cos t$

$= \underline{\underline{e^{-2t}(5\sin t - 2\cos t)}}$

e) $f(s) = \dfrac{0.9s}{(s-0.1)(s+0.2)} = \dfrac{A}{(s-0.1)} + \dfrac{B}{(s+0.2)} = \dfrac{A(s+0.2)+B(s-0.1)}{(s-0.1)(s+0.2)}$

s^1: $A+B = 0.9$
s^0: $0.2A - 0.1B = 0$

$\Rightarrow$ $A = 0.3$
$B = 0.6$

$f(s) = \dfrac{0.3}{(s-0.1)} + \dfrac{0.6}{(s+0.2)}$

$\mathcal{L}^{-1}\{f(s)\} = 0.3\mathcal{L}^{-1}\left\{\dfrac{1}{(s-0.1)}\right\} + 0.6\mathcal{L}^{-1}\left\{\dfrac{1}{(s+0.2)}\right\}$

$= \underline{\underline{0.3e^{(0.1)t} + 0.6e^{-(0.2)t}}}$

f) $f(s) = \dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4}$

$$= \frac{As(s^2+4) + B(s^2+4) + Cs(s^2+1) + D(s^2+1)}{(s^2+1)(s^2+4)}$$

s^3: $A+C=0$
s^2: $B+D=0$
s^1: $4A+C=0$
s^0: $4B+D=1$

The 1st and 3rd equations imply $A=C=0$
The 2nd and 4th equations imply $B = 1/3$ and $D = -1/3$

$\therefore f(s) = \dfrac{1/3}{s^2+1} + \dfrac{-1/3}{s^2+4}$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \underline{\underline{\tfrac{1}{3}\sin t - \tfrac{1}{6}\sin 2t}}$$

Problem 3

a) $y' - y = 1$, $y(0) = 1$, $y = y(t)$

Begin by taking the Laplace transform of both sides

$$\mathcal{L}\{y' - y\} = \mathcal{L}\{1\}$$

$$\Rightarrow \mathcal{L}\{y'\} - \mathcal{L}\{y\} = \frac{1}{s}$$

The Laplace transform of a derivative is given by

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0)$$

So,

$$s\mathcal{L}\{y(t)\} - y(0) - \mathcal{L}\{y(t)\} = 1/s \quad \text{and } y(0)=1$$

$$(s-1)\,\mathcal{L}\{y(t)\} = \frac{1}{s} + 1$$

$$\mathcal{L}\{y(t)\} = \frac{1}{s(s-1)} + \frac{1}{s-1} = \frac{-1}{s} + \frac{2}{s-1}$$ by a partial fraction decomposition

$$\therefore\; y(t) = \mathcal{L}^{-1}\left\{\frac{-1}{s} + \frac{2}{s-1}\right\}$$ after inverting the transform

$$= -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \underline{\underline{-1 + 2e^{t}}}$$

b) $y' + 6y = e^{4t}, \quad y(0) = 2$

$$\mathcal{L}\{y'+6y\} = \mathcal{L}\{y'\} + 6\mathcal{L}\{y\}$$

$$= s\mathcal{L}\{y(t)\} - y(0) + 6\mathcal{L}\{y(t)\}$$

$$= (s+6)\mathcal{L}\{y(t)\} - 2$$

$$\mathcal{L}\{e^{4t}\} = \frac{1}{s-4}$$

$$\therefore\; (s+6)\,\mathcal{L}\{y(t)\} - 2 = \frac{1}{s-4}$$

$$\Rightarrow\; \mathcal{L}\{y(t)\} = \frac{1}{10(s-4)} - \frac{19}{10(s+6)}$$ after a partial fraction decomposition on the RHS

So,

$$y(t) = \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \frac{19}{10}\mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\}$$

$$= \frac{1}{10}e^{4t} - \frac{19}{10}e^{-6t}$$

c) $y'' + 5y' + 4y = 0 \quad , \; y(0) = 1 \, , \; y'(0) = 0$

$$\mathcal{L}\{y'' + 5y' + 4y\} = \mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\}$$

$$= s^2\mathcal{L}\{y(t)\} - sy(0) - \cancel{y'(0)}^{\to 0} + 5s\mathcal{L}\{y(t)\} - 5y(0) + 4\mathcal{L}\{y(t)\}$$

$$= (s^2 + 5s + 4)\mathcal{L}\{y(t)\} - (s+5)$$

$$\mathcal{L}\{0\} = 0$$

$$\therefore \quad \mathcal{L}\{y(t)\} = \frac{s+5}{s^2+5s+4} = \frac{s+5}{(s+1)(s+4)} = \frac{4}{3(s+1)} - \frac{1}{3(s+4)}$$

So, $y(t) = \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$

d) $y''' + 2y'' - y' - 2y = \sin(3t) \quad , \; y(0) = 0, \; y'(0) = 0, \; y''(0) = 1$

$$\mathcal{L}\{y'''\} + 2\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\}$$

$$= s^3\mathcal{L}\{y(t)\} - y''(0) - \cancel{sy'(0)}^{\to 0} - \cancel{s^2y(0)}^{\to 0} + 2s^2\mathcal{L}\{y(t)\} - 2\cancel{y'(0)}^{\to 0}$$

$$- \cancel{2sy(0)}^{\to 0} - s\mathcal{L}\{y(t)\} + \cancel{y(0)}^{\to 0} - 2\mathcal{L}\{y(t)\}$$

$$= (s^3 + 2s^2 - s - 2)\mathcal{L}\{y(t)\} - 1$$

$$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$$

$$\therefore \quad (s^3+2s^2-s-2)\mathcal{L}\{y(t)\} - 1 = \frac{3}{s^2+9}$$

$$\mathcal{L}\{y(t)\} = \frac{3}{(s^2+9)(s^3+2s^2-s-2)} + \frac{1}{(s^3+2s^2-s-2)}$$

$$= \frac{3}{(s^2+9)(s-1)(s+1)(s+2)} + \frac{1}{(s-1)(s+1)(s+2)}$$

which should give after a partial fraction decomposition

$$= \frac{\left(\frac{3}{130}s - \frac{9}{195}\right)}{s^2+9} + \frac{13/60}{s-1} - \frac{13/20}{s+1} + \frac{16/39}{s+2}$$

$$\Rightarrow y(t) = \frac{3}{130}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{3}{195}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} + \frac{13}{60}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$-\frac{13}{20}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{16}{39}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \underline{\frac{3}{130}\cos 3t - \frac{3}{195}\sin 3t + \frac{13}{60}e^{t} - \frac{13}{20}e^{-t} + \frac{16}{39}e^{-2t}}$$