Problem 4

a) $\Gamma(\alpha+1) = \int_0^\infty e^{-t}\, t^\alpha\, dt \qquad \alpha > 0$

integrating by parts

$$= \cancelto{0}{t^\alpha(-e^{-t})\Big|_0^\infty} - \int_0^\infty (-e^{-t})\,\alpha\, t^{\alpha-1}\, dt$$

$$= \alpha \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt$$

$$= \alpha\,\Gamma(\alpha)$$

b) $\mathcal{L}\{t^\alpha\} = \int_0^\infty e^{-st}\, t^\alpha\, dt$

$$= \int_0^\infty e^{-st}\, \frac{(st)^\alpha}{s^\alpha}\, \frac{dt}{d(st)}\, d(st) \qquad \frac{dt}{d(st)} = \frac{1}{s}$$

$$= \frac{1}{s^{\alpha+1}} \int_0^\infty e^{-st}\, (st)^\alpha\, d(st)$$

$$= \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

Problem 5

$\Gamma(1/2) = \sqrt{\pi}$

a) $\mathcal{L}\{t^{-1/2}\} = \dfrac{\Gamma(-1/2+1)}{s^{(-1/2+1)}} = \sqrt{\dfrac{\pi}{s}}$

b) $\mathcal{L}\{t^{1/2}\} = \dfrac{\Gamma(\frac{1}{2}+1)}{s^{(1/2+1)}} = \dfrac{1/2\,\Gamma(1/2)}{s^{3/2}} = \dfrac{1}{2}\sqrt{\dfrac{\pi}{s^3}}$

c) $\mathcal{L}\{t^{3/2}\} = \dfrac{\Gamma(3/2+1)}{s^{(3/2+1)}} = \dfrac{3/2\,\Gamma(3/2)}{s^{5/2}} = \dfrac{3/2\left(\frac{1}{2}\Gamma(\frac{1}{2})\right)}{s^{5/2}} = \dfrac{3}{4}\sqrt{\dfrac{\pi}{s^5}}$

Problem 6

$$f(s) = \frac{k^2}{s(s^2+k^2)} = \frac{1}{s} - \frac{s}{s^2+k^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+k^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$$
$$= 1 - \cos kt$$

$$\therefore F(t) = 1 - \cos kt$$