4. a) $\mathcal{F}\{u(x,t)\} = \mathcal{F}\{\frac{1}{2\sqrt{k\pi t}} \int_{-\infty}^{\infty} f(\tau) e^{-(x-\tau)^2/4kt} d\tau$

$= U(\alpha,t) = \mathcal{F}\{f(x)\} \cdot \mathcal{F}\{\frac{1}{2\sqrt{k\pi t}} e^{-x^2/4kt}\}$

$\therefore p = \sqrt{kt}$

$= F(\alpha) \cdot \frac{2\sqrt{\pi k t}}{2\sqrt{k\pi t}} e^{-kt\alpha^2} = F(\alpha) e^{-kt\alpha^2}$

$U(\alpha, 0) = F(\alpha) \Rightarrow \mathcal{F}^{-1}(U(\alpha,0)) = \mathcal{F}^{-1}\{F\alpha$

$= f(x) = u(x,0)$

thus, $u(x,0) = f(x)$ condition is satisfied

Taking $k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

$\mathcal{F}\{k \frac{\partial^2 u}{\partial x^2}\} = \mathcal{F}\{\frac{\partial u}{\partial t}\}$

$= -k\alpha^2 U(\alpha,t) = \frac{dU}{dt}$ ①

Taking $U(\alpha,t) = F(\alpha) e^{-kt\alpha^2}$ from before, we see that

$\frac{dU(\alpha,t)}{dt} = -k\alpha^2 U(\alpha,t)$. Thus, equation ① is satisfied.

$\Rightarrow u(x,t) = \frac{1}{2\sqrt{k\pi t}} \int_{\infty} f(\tau) e^{-(x-\tau)^2/4kt} d\tau$

satisfies both $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ and

$u(x,0) = f(x)$

$\Rightarrow u(x,t)$ is a solution.

b) let $u = (x-\tau)/2\sqrt{kt}$ then

$\frac{du}{d\tau} = -\frac{1}{2\sqrt{kt}}$, $\tau = (x - u \cdot 2\sqrt{kt})$

$u(x,t) = \frac{1}{2\sqrt{k\pi t}} \int_{-\infty}^{\infty} f(\tau) e^{-u^2} \cdot (-2\sqrt{kt}) du$

$= -\frac{1}{\sqrt{\pi}} \int_{\frac{1+x}{2\sqrt{kt}}}^{\frac{1-x}{2\sqrt{kt}}} u_0 e^{-u^2} du$

$\therefore |\tau| < 1 \Rightarrow -1 < x - u \cdot 2\sqrt{kt} < 1$

$\Rightarrow -1 < u \cdot 2\sqrt{kt} - x < 1 \Rightarrow \frac{-1+x}{2\sqrt{kt}} < u < \frac{1+x}{2\sqrt{kt}}$

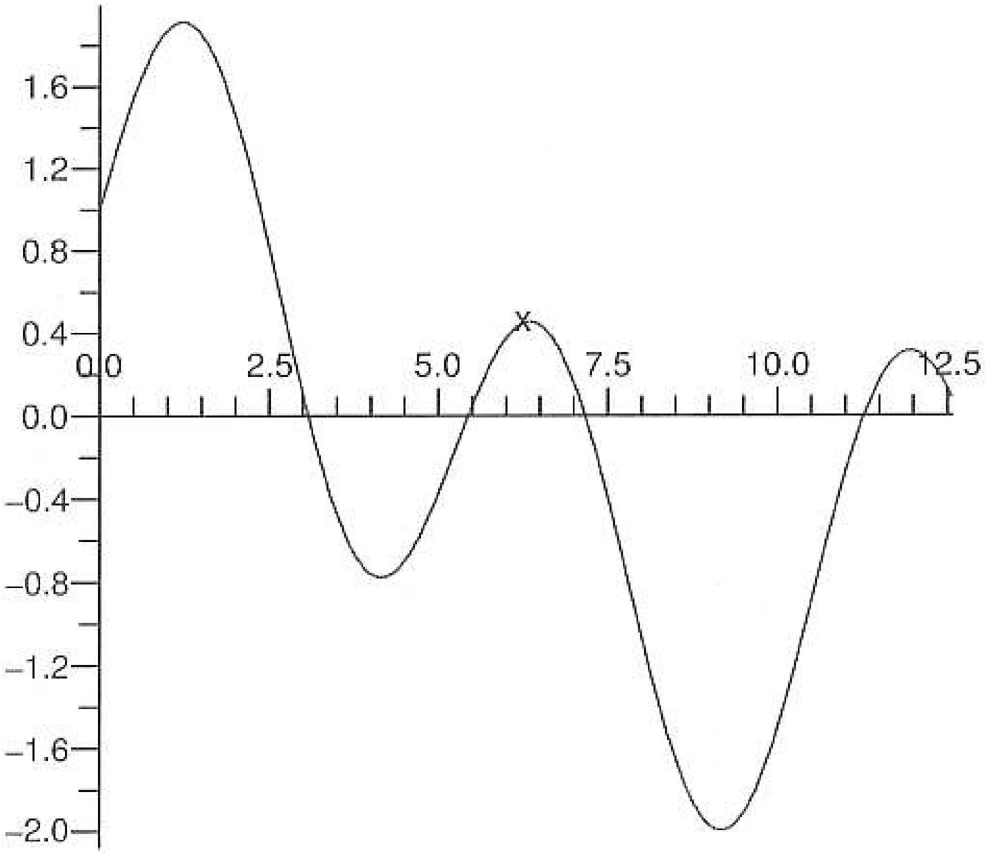
$\Rightarrow = \frac{1}{2} u_0 \left(\operatorname{erf}\left(\frac{1+x}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{1-x}{2\sqrt{kt}}\right) \right)$

5.

$f := \sin(1.2 \cdot x) + \cos\left(\frac{x}{3}\right);$

$$\sin(1.2\, x) + \cos\left(\frac{1}{3}\, x\right) \tag{1}$$

plot(*f*, *x* = 0 ..(4·Pi));



with(*inttrans*);

[*addtable*, *fourier*, *fouriercos*, *fouriersin*, *hankel*, *hilbert*, *invfourier*, *invhilbert*, *invlaplace*, *invmellin*, *laplace*, *mellin*, *savetable*] (2)

fourier(*f*, *x*, *s*);

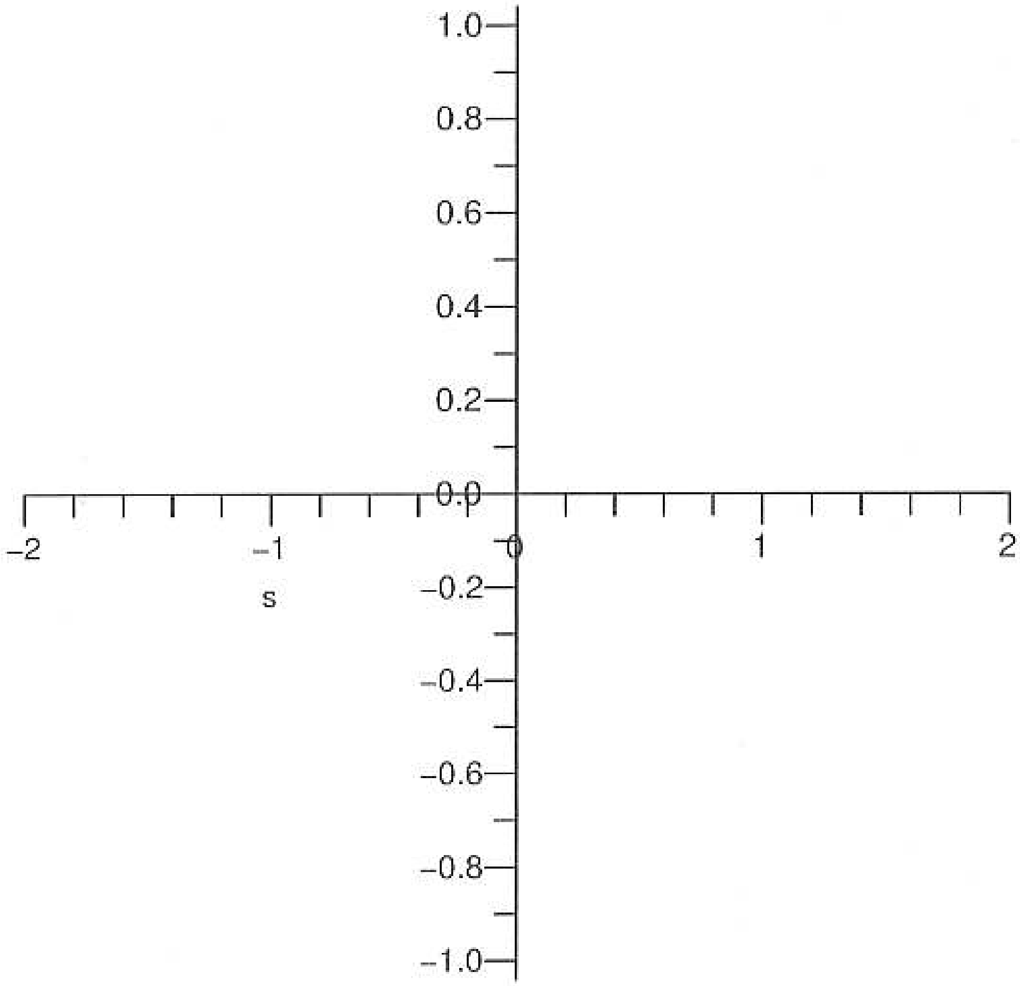
$$3.141592654\, \mathrm{I}\, \mathrm{Dirac}(s + 1.200000000) + 3.141592654\, \mathrm{Dirac}(s - 0.3333333333) - 3.141592654\, \mathrm{I}\, \mathrm{Dirac}(s - 1.200000000) + 3.141592654\, \mathrm{Dirac}(s + 0.3333333333) \tag{3}$$

abs(%);

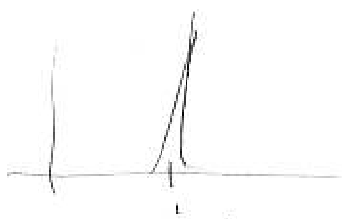
$$|3.141592654\, \mathrm{I}\, \mathrm{Dirac}(s + 1.200000000) \tag{4}$$

$$+ 3.141592654\,\mathrm{Dirac}(s-0.3333333333)$$
$$-3.141592654\,\mathrm{I}\,\mathrm{Dirac}(s-1.200000000)$$
$$+ 3.141592654\,\mathrm{Dirac}(s+0.3333333333)$$

$plot(\%, s=-2..2);$



Spikes not shown because Maple refuses to. Dirac is the δ you normally see. it is shaped like the following usually



$\delta(s-1)$

6. Note: there is a typo in the problem set. We solve 15.5.2

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = \int_{a-\varepsilon}^{a+\varepsilon} \frac{1}{2\varepsilon} f(x)\,dx$$

According to the mean value theorem,

$$\frac{1}{2\varepsilon}\cdot 2\varepsilon \underbrace{\min_{a-\varepsilon<x<a+\varepsilon}}_{\text{interval from } a-\varepsilon \text{ to } a+\varepsilon} (f(x)) \le \int_{a-\varepsilon}^{a+\varepsilon} \frac{1}{2\varepsilon} f(x)\,dx \le \frac{2\varepsilon}{2\varepsilon} \max(f$$

$$\Rightarrow \min(f(x)) \le \int_{a-\varepsilon}^{a+\varepsilon} \frac{1}{2\varepsilon} f(x)\,dx \le \max(f(x$$

as $\varepsilon \to 0$, $\min_{a-\varepsilon<x<a+\varepsilon}(f(x)) = \max(f(x)) = f(a)$

$$\Rightarrow \min f(x) \le \int_{a-\varepsilon}^{a+\varepsilon} \frac{1}{2\varepsilon} f(x)\,dx \le \max(f(x))$$

$\min f(x) \to f(a)$, $\int_{a-\varepsilon}^{a+\varepsilon} \frac{1}{2\varepsilon} f(x)\,dx \to f(a)$ as well by squeeze theorem, $\max(f(x)) \to f(a)$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$