Statistical Physics, Phys504b/AM531b. Winter/spring 2008

Problem set 6, Mar. 31, 2008.

Please write out all the details in your calculations.

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1. **The Ising model.** Diagonalize explicitly ($\equiv$ *all* steps of the calculation must be shown) the 1D Ising model transfer matrix T and find its eigenvalues.

2. **Microcanonical ensemble.** Assume that we have a relativistic ideal gas described by the Hamiltonian
$$\mathcal{H}(p,q) = \sqrt{p^2c^2 + m^2c^4} - mc^2,$$
where c is the speed of light. Assume the microcanonical ensemble and compute $S(E,V)$, $E(T,V)$ and $p(T,V)$. You can assume that $v/c \approx 1$. Determine also the equation of state.

3. **Microcanonical ensemble.** Assume the the classical ideal gas in the microcanonical ensemble. Derive (full derivation using the distribution function) the Sakur-Tetrode equation
$$S = k_B N \left[\ln \left\{ \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} \frac{V}{N} \right\} + \frac{5}{2} \right] + \text{higher order corrections.}$$
Compute also the leading correction terms.

4. **Canonical ensemble.** Assume an classical ideal gas in the canonical ensemble. Derive (*full* derivation using the distribution function) the Sakur-Tetrode equation
$$S = k_B N \left[\ln \left\{ \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} \frac{V}{N} \right\} + \frac{5}{2} \right] + \text{higher order corrections.}$$
Compute also the leading correction terms.

5. **Phase space volume.** Show that the volume of an N-dimensional hyper-sphere with radius R is given by
$$V_N(R) = \frac{\pi^{N/2}}{(N/2)\Gamma(N/2)} R^N,$$
where $\Gamma(N) = (N-1)!$. Hint: Use the fact that $\int_{-\infty}^{-\infty} dy \exp(-\alpha y^2) y^N$ can be expressed in terms of the Γ-function.

6. **Harmonic oscillators.** Assume that we have N harmonic oscillators (classical) with the Hamiltonian (for an oscillator)
$$H(p,q) = \frac{p^2}{2m} + \frac{kq^2}{2}, \text{where} k = m\omega^2.$$
Calculate (all details!) $S(E)$, and $E(N,T)$ and $C(N,T)$. (Hint: By changing scales, the surface of constant energy can be deformed into a sphere. This can also be done using a canonical transformation).

7. **Quantum harmonic oscillators.** Consider N independent quantum oscillators subject to a Hamilotnian

$$H(\{n_i\}) = \sum_{i=1}^{N} \hbar\omega \left(n_i + \frac{1}{2} \right),$$

where $n_i = 0, 1, 2, ...$ is the quantum occupation number for the ith oscillator. Calculate $S(E)$ and $E(T, N)$. Find the probability that a particular oscillator is in its nth level,

8. **Harmonic oscillators: Grand canonical ensemble.** Assume that we have N harmonic oscillators (classical) with the Hamiltonian (for an oscillator)

$$H(p, q) = \frac{p^2}{2m} + \frac{kq^2}{2},$$

and assume the grand canonical ensemble. Derive (all details!) expression for the partition function, Ω, S, and N.

9. **Equipartition.** The equipartition theorem for classical systems can be written in a general from as

$$\langle p_i \frac{\partial H}{\partial p_i} \rangle = \langle q_i \frac{\partial H}{\partial q_i} \rangle = k_B T$$

for each component p_i of $\vec{p}$. Show that this holds using the canonical ensemble (show all details of your calculation!!). Hint: Use partial intregratin and assume "infinite walls"at $\pm\infty$.