On the Concept of Local Fractional Differentiation

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Abstract

The concept of local fractional derivative is studied. A more straightforward definition is proposed and its properties are studied. This local fractional derivative is applied to stable distributions. The possibility of defining local fractional derivative based on the Weyl derivative is examined. The relationship between the resulting local fractional derivative and the Kolwankar-Gangal derivative is established.

1 Introduction

Non-integral order derivation has been attempted ever since the birth of calculus itself [9]. The basis for defining fractional derivatives is the relationship between the integer n and the n^{th} order derivatives. A remarkable merit of these fractional differentiation operators is that they may still apply to functions which are not differentiable in the classical sense.

However, the domain and boundary conditions of the functions must be explicitly chosen and restricted in order to define differentiation as a linear operator. Information about the domain and boundary condition of the functions affect the fractional order derivatives. Therefore, unlike the integral order derivative, the fractional order derivative at a point x is not determined by an arbitrarily small neighborhood of x. In other words, the fractional derivative is not a local property of the function. This obscures the geometric meaning of these fractional derivatives. In [4] the Kolwankar-Gangal (henceforth K-G) local fractional derivative is defined via:

$$D^q f(x_0) = \lim_{x \to x_0} \frac{d^q(f(x) - f(x_0))}{d(x - x_0)^q}, \tag{1}$$

where the operator in the RHS is the Riemann-Liouville derivative. Here we investigate the properties of a new local fractional derivative defined based on the Weyl fractional derivative.

In [5] a definition of local fractional differentiation via Fourier series is proposed. The approach is: For any x_0, take a local copy of the function $f(x)$ centered at x_0, perform some modification, and compute the Weyl fractional derivative of this modified function. Some properties of this derivative are given, but the computation is complicated and its relationship with the K-G derivative is not clear.

In this paper we present a new approach to defining local fractional derivative using the Weyl fractional integral and derivative. This definition has some nice properties. We present a result here which shows that it is fairly easy to compute. We show that although it is different from the K-G derivative, they have some interesting connections.

2 K-G Local Fractional Derivative

From the definition of R-L derivative, the K-G local fractional derivative is essentially

$$\left.\frac{d^\alpha}{dx^\alpha} f(y)\right|_{x=y} = \frac{1}{\Gamma(1-\alpha)} \lim_{x\to y} \frac{d}{dx} \int_y^x (x-t)^{-\alpha}(f(t)-f(y))dt. \tag{2}$$

It is the limit of the first order derivative of the fractional integral

$$I(x) = \int_y^x (x-t)^{-\alpha}(f(t)-f(y))dt \tag{3}$$

when $x \to y$, i.e. $\lim_{x\to y} I'(x)$.

One may wonder why it is not defined directly as the derivative of $I^\alpha(x)$ at y

$$I'(y). \tag{4}$$

In fact, in [4] (4) was used implicitly. However, obviously these two are not completely equivalent. The necessary and sufficient condition for both (4) and (1) to exist and to be equal is that $I(x)$ be continuously differentiable.

We have several reasons to believe that (4) serves as a better definition for local fractional derivative. First, it has a better chance to exist since it merely requires the existence of the derivative of $I'(y)$, while K-G derivative requires the existence of $I'(x)$ for $x > y$ and the limit $\lim_{x\to y} I'(x)$. Second, suppose that $I'(y)$ does not exist, then even if $I'(x)$ were to exist for $x > y$, and if the limit as $x \to y$ were to exist, this limit would not represent the local behavior of $I(x)$ at y, and hence could lead to incorrect behavior of fractional derivative of $f(x)$ at y. In fact, in [4] (4) was implicitly used in the development of a fractional Taylor series for detecting the local singularity of fractal functions. Third, as we will show now, (4) has a very straightforward equivalent form, which makes its calculation much easier even for complicated fractal functions.

Proposition 1.

$$I'(y) = \Gamma(1+\alpha) \lim_{h\to 0+} \frac{f(y+h)-f(y)}{h^\alpha}. \tag{5}$$

Proof. Since $f(x)$ is continuous in a neighborhood of y, we can assume that $|f(x)| \leq M$, therefore

$$|I(x)| \leq \int_y^x (x-t)^{-\alpha} M dy = M\,|x-y|^{1-\alpha} \rightsquigarrow I(y) = 0 \tag{6}$$

Hence we have

$$\begin{aligned} \frac{I(y+h)-I(y)}{h} &= \frac{1}{h}\int_y^{y+h} (y+h-t)^{-\alpha}(f(t)-f(y))dt \\ &= \int_0^1 (1-t)^{-\alpha}\frac{f(ht+y)-f(y)}{h^\alpha}dt. \end{aligned} \tag{7}$$

Let

$$\begin{aligned} k_1 &= \limsup_{h\to 0} \frac{f(\epsilon+y)-f(y)}{h^\alpha}, \\ k_2 &= \liminf_{h\to 0} \frac{f(\epsilon+y)-f(y)}{h^\alpha}. \end{aligned} \tag{8}$$

Since $t < 1$, then for any $\delta < 0$, h sufficiently small,

$$\begin{aligned} \frac{I(y+h)-I(y)}{h} &< (k_1+\delta)\int_0^1 (1-t)^{-\alpha}t^\alpha dt, \\ \frac{I(y+h)-I(y)}{h} &> (k_2-\delta)\int_0^1 (1-t)^{-\alpha}t^\alpha dt. \end{aligned} \tag{9}$$

We see that $I'(y)$ exists if and only if $k_1 = k_2 = k$, and in which case we have

$$\lim_{h\to 0} \frac{I(y+h)-I(y)}{h} = k\int_0^1 (1-t)^{-\alpha}t^\alpha dt = \Gamma(1+\alpha)k. \tag{10}$$

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With this conclusion, it is easy to see that the condition for a function to be locally α-differentiable at x_0 is

$$f(x) - f(x_0) = A\,|x-x_0|^\alpha + o(|x-x_0|^\alpha), \quad x - x_0 \to 0. \tag{11}$$

Now we use the above as the new definition of local fractional derivative.

Definition 1. *Define the right and left local fractional derivative by*

$$\begin{aligned} D_+^\alpha f(y) &= \Gamma(1+\alpha)\lim_{h\to 0+} \frac{f(y+h)-f(y)}{h^\alpha}, \\ D_-^\alpha f(y) &= -\Gamma(1+\alpha)\lim_{h\to 0+} \frac{f(y-h)-f(y)}{h^\alpha}. \end{aligned} \tag{12}$$

If $D_+^\alpha f(y) = D_-^\alpha f(y)$ we say function f is locally α-differentiable at y and denote the common value by

$$D^\alpha f(y). \tag{13}$$

For $\alpha > 1$, let $\alpha = n + \beta$, where $n \geq 1$ is integer and $0 < \beta < 1$. We call f α-differentiable at y if the n^{th} derivative of f near y exists and $f^{(n)}$ is β-differentiable at y.

Proposition 2. *For $0<\alpha<1$, f is α-differentiable at y if and only if*

$$f(x+h)=f(x)+sign(h)A\left|h\right|^{\alpha}\qquad h\to 0. \tag{14}$$

For $\alpha=n+\beta>1$, f is α-differentiable if and only if

$$f(x+h)=\sum_{k=0}^{n-1}f^{(k)}(x)h^{k}+sign(h^{n+1})A\left|h\right|^{\alpha},\quad h\to 0. \tag{15}$$

Obviously for $0<\alpha<1$ the function x^{α} has non-trivial α-derivative, but only at a single point $x=0$. If $f(x)$ has a non-trivial α-derivative at a point x_0, its graph must be smooth at $x=x_0$ but with a vertical tangent line. One can quickly conclude that it is not possible for a function to possess a continuous non-trivial α-fractional derivative on an interval. It is also true that the α-th derivative for any smooth function must be zero. In fact, if a function belongs to H^{γ} then for any $\alpha<\gamma$, the α-derivative vanishes. For $\alpha>\gamma$, the α-derivative could be divergent. Therefore in order to find the usefulness of local fractional derivative we would naturally turn to functions which are nowhere differentiable, i.e., functions whose maximal Hölder order is less than 1. Unfortunately, the existence of the local fractional derivative is not common, at least among the favorite examples of fractal geometry.

For example, consider the function of the Devil's staircase. The Devil's staircase refers to a function $f(x)$ that is continuous and nondecreasing on $[a,b]$, with $f(a)<f(b)$ and $f'(x)=0$ outside a zero-measure set. One staircase is the cumulative distribution function of the random variable that is uniformly distributed on the Cantor set. This function is possible to have a local fractional derivative only for order $\alpha=\log_3 2$. For $\alpha=\log_3 2$, for any x in the Cantor set,

$$\limsup_{h\to 0}\frac{f(\epsilon+y)-f(y)}{h^{\alpha}}=1\neq\liminf_{h\to 0}\frac{f(\epsilon+y)-f(y)}{h^{\alpha}}=2^{-\alpha}. \tag{16}$$

Therefore the α derivative does not exist.

Now we give a method to construct functions which have non-trivial α-derivative on a fractal set.

First a function $f(x)$ on $[a,b]$ is said to have Property B if it satisfies the following

1. $f(x)$ is smooth and monotonously increasing on (a,b).
2. For $0<\alpha<1$, $f(x)$ behaves like x^{α} near a and b

$$\begin{aligned}f(x)&=f(a)+(x-a)^{\alpha}+o((x-a)^{\alpha}),x\to a,\\ f(x)&=f(b)-(b-x)^{\alpha}+o((b-x)^{\alpha}),x\to b.\end{aligned} \tag{17}$$

Let $\Delta=\frac{b-a}{3}$, $c=a+\Delta$, $d=a+2\Delta$. Next for any $\epsilon>0$, $\delta>0$ we perform a transform $T_{\delta,\epsilon}$ on $f(x)$ such that $g(x)=T_{\delta,\epsilon}f(x)$ and both $g(x)$, $x\in[a,c]$ and $g(x)$, $x\in[d,b]$ have Property B, and

$$g(x)=f(x),x\in[a,c-\delta]\cup[c+\delta,d-\delta]\cup[d+\delta,b] \tag{18}$$

$$|g(x)-f(x)|\leq\epsilon,\quad x\in[a,b] \tag{19}$$

Obviously such a transform is always possible for any $\delta > 0$ and $\epsilon > 0$. Now we start with an B-function $g_0(x)$ defined on $[0,1]$. First we perform transform T_{δ_0,ϵ_0} on $g_0(0,1)$ to get g_1. Next we perform transform T_{δ_1,ϵ_1} on $g_1(0,1/3)$ and $g_1(2/3,1)$ respectively to get g_2, and go on recursively on each resulting B-function. Note that $\Delta_n = 3^{-n}$. We choose δ and ϵ such that

$$\frac{\delta}{\Delta_n} \to 0, \quad \frac{\epsilon}{\Delta_n^\alpha} \to 0, \quad n \to \infty. \tag{20}$$

Note that no part of the original function g_0 undergoes alteration more than once. Since $|g_n(x) - g_{n+1}(x)| \le \epsilon < 3^{-n}$, by Cauchy's convergence theorem this function sequence uniformly converges to a limiting function $g(x)$. For each x in the Cantor set, at some point during the transform process, say kth step, x will become the left- or right-end point of an B-function. The α-derivative of g_k at x is obviously 1. Subsequent transformations will make changes to g_k on a series of 2δ-wide intervals. We only need to prove that when $h \to 0$ and $x + h$ falls inside these intervals the ratio $\frac{f(x+h)-f(x)}{h^\alpha}$ is still 1. This has been guaranteed by (20). Hence the limiting function $g(x)$ has α-derivative with value 1 at each point of the Cantor set. Outside the Cantor set $g(x)$ is actually smooth, since for each x, when n is large enough, g_n will remain unchanged as a smooth function in an interval containing x. Note that the number α is not related to the dimension of the Cantor set.

3 Stable Distributions

It is well known that if the characteristic function of a probability distribution is second order differentiable at the origin, then this distribution belongs to the domain of attraction of the Gaussian distribution. This is one way of stating the central limiting theorem. Since we have defined the local α-derivative of a function, we will show that a similar conclusion is true for characteristic functions which are α-differentiable at 0, where $0 < \alpha < 2$.

Proposition 3. *Suppose $\phi(t)$ is the characteristic function of a certain probability distribution, then if for $0 < \alpha < 2$, $\alpha \ne 1$, ϕ is α-differentiable at 0, then the distribution belongs to the domain of attraction of a symmetric stable distribution.*

Proof. Let $X_1, X_2, \cdots X_n$ be n independent random variables governed by the distribution.

If $\alpha > 1$, $\phi(t)$ is differentiable at 0, therefore the mean of X exists. Let μ be the mean of X_k and consider

$$Z_n = n^{-\frac{1}{\alpha}} \sum_{k=1}^{n} (X_k - \mu), \tag{21}$$

for which the characteristic function is

$$\phi_n(t) = \left[e^{-\mu n^{-\frac{1}{\alpha}} t} \phi(n^{-\frac{1}{\alpha}} t) \right]^n. \tag{22}$$

Since ϕ is α-differentiable at 0, $1 < \alpha < 2$, $n = 1$ and $\text{sign}(h^{n+1}) > 0$. Then

$$\phi(t) = 1 + \mu t + A\,|t|^{\alpha} + o(|t|^{\alpha}), \quad t \to 0. \tag{23}$$

Considering $e^{-\mu t} = 1 - \mu t + o(t^2)$ we have

$$\begin{aligned} \phi_n(t) &= \left[1 + A(n^{-\frac{1}{\alpha}}\,|t|)^{\alpha} + o\left|(n^{-\frac{1}{\alpha}})t\right|^{\alpha}\right]^n, \quad n \to \infty \\ &= \left[1 + \frac{A\,|t|^{\alpha} + o\,|t|^{\alpha}}{n}\right]^n, \quad n \to \infty \\ &\to e^{A|t|^{\alpha}} = e^{-\sigma^{\alpha}|t|^{\alpha}}, \quad n \to \infty \end{aligned} \tag{24}$$

which converges for each fixed t. That is, the limit of the characteristic function of Z_n is the characteristic function of a symmetric stable distribution [10].

For $\alpha < 1$ we consider the random variable

$$Z_n = n^{-\frac{1}{\alpha}} \sum_{k=1}^{n} X_k. \tag{25}$$

Its characteristic function is

$$\phi_n(t) = \left[\phi(n^{-\frac{1}{\alpha}}t)\right]^n, \tag{26}$$

and the rest of the proof is identical. □

From this we see that both the local and the conventional fractional derivatives has interesting connections with the stable distributions The connection between the conventional fractional derivative and the stable distribution has been noted by many authors. (e.g., in [2]).

4 Weyl Fractional Integral and Derivative

In this section we state some facts about the the Weyl fractional integral and derivative. They can be found in [9, 11].

A function $f(x)$ is said to satisfy the *Hölder condition of order* λ on a set Ω if

$$|f(x) - f(y)| \le A\,|x - y|^{\lambda}, \quad \forall x, y \in \Omega \tag{27}$$

The space of all such functions is denoted by $H^{\lambda}(\Omega)$.

Let $f(x)$ be a 2π-periodic function on $\mathbb{R}$ with zero mean, i.e.,

$$2\pi f_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = 0. \tag{28}$$

Let

$$f(x) \sim \sum_{-\infty}^{\infty} f_k e^{-ikx}, \quad f_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)dx, \tag{29}$$

be its Fourier series. Since $f^{(n)}(x) \sim \sum_{-\infty}^{\infty}(ik)^n f_k e^{ikx}$, Weyl defined fractional integration and differentiation in the following way

$$D^\alpha f \sim \sum_{-\infty}^{\infty}(ik)^\alpha f_k e^{ikx}, \quad I^\alpha f \sim \sum_{-\infty}^{\infty}(ik)^{-\alpha} f_k e^{ikx}. \tag{30}$$

The fractional integral can be expressed as a convolution

$$I^\alpha f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-t)\Psi_\alpha(t)dt, \tag{31}$$

where

$$\Psi_\alpha(t) = \sum_{k=-\infty, k\neq 0}^{\infty} \frac{e^{ikt}}{(ik)^\alpha}. \tag{32}$$

For $0 < \alpha < 1$, the function Ψ has the form

$$\Psi_\alpha(t) = \frac{2\pi}{\Gamma(1-\alpha)} t_+^{\alpha-1} + r_\alpha(t), \quad -2\pi < t \leq 2\pi \tag{33}$$

where the function $r_\alpha(t)$ is infinitely differentiable for $t \in (-2\pi, 2\pi]$. $t_+ = t$ for $t \geq 0$ and 0 for $t < 0$. Therefore, Ψ' satifies the following

$$\left|\Psi'_\alpha(t)\right| \leq C_\alpha \left|t\right|^{\alpha-1}. \tag{34}$$

Another way of defining fractional derivative is via fractional integration and conventional differentiation

$$D^\alpha f = \frac{d}{dx} I^{1-\alpha} f. \tag{35}$$

If $f \in H^\lambda[0, 2\pi]$ and $\alpha < \lambda$, then Weyl fractional derivative can also be written as

$$D^\alpha f = \frac{1}{2\pi}\left[f(x-t) - f(x)\right]\Psi'_{1-\alpha}(t)dt. \tag{36}$$

5 The Approach via Weyl Fractional Derivative

In order to use the Weyl derivative to define a local fractional derivative, naturally, for any small $\epsilon > 0$ we would consider $f(x)$ restricted on $[x_0, x_0 + 2\epsilon]$ and extend it to a 2ϵ-periodic function. However, such a function may not even be continuous, let alone to have other properties, since the two ends need not meet. Therefore we should perform some modification on the restriction of $f(x)$ such that it can be extended to a 2ϵ periodic function. This modification is delicate since it must resemble the original function as much as possible such that the difference will not affect the final result when letting $\epsilon \to 0$.

Definition 2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function,* $x_0 \in \mathbb{R}$, *and* $\epsilon > 0$. *Let function* $f_{x_0,\delta,\epsilon} : [0, 2\pi] \to \mathbb{R}$ *be continuous and satisfy*

1.

$$f_{x_0,\delta,\epsilon}(0) = f_{x_0,\delta,\epsilon}(2\pi). \tag{37}$$

2.

$$f_{x_0,\delta,\epsilon}(x) = f\left(\frac{\delta}{\pi}x + x_0\right) - c, \quad 0 < x < 2\pi - 2\epsilon. \tag{38}$$

where c is a constant.

3.

$$\int_0^{2\pi} \left| f_{x_0,\delta,\epsilon}(x) - f\left(\frac{\delta}{\pi}x + x_0\right) + c \right| dx < \epsilon. \tag{39}$$

4. $f_{x_0,\delta,\epsilon}(x)$ on $[0, 2\pi]$ and f on $[x_0, x_0+2\delta]$ share the same Hölderian characteristic.

Let $0 < \alpha \le 1$. For any $\theta \in (0, 2\pi - 2\epsilon)$, suppose that as $\delta \to 0$ and $\epsilon \to 0$, $\frac{\epsilon}{\delta^\alpha} \to 0$, then define

$$D_\theta^\alpha f(x_0) = \lim_{\delta,\epsilon \to 0} \left(\frac{\pi}{\delta}\right)^\alpha D^\alpha f_{x_0,\delta,\epsilon}(\theta), \tag{40}$$

where $D^\alpha f_{x_0,\delta,\epsilon}(\theta)$ is the value of the Weyl fractional derivative of the 2π-periodic function $f_{x_0,\delta,\epsilon}$ at $\theta \in (0, 2\pi)$.

The idea of this definition is taking a local copy of the function, making it periodic while keeping the difference with the original function as small as possible. In order to justify this process we must guarantee both that the function $f_{x_0,\delta,\epsilon}$ exists and that the existence and value of the limit are independent of the selection of $f_{x_0,\delta,\epsilon}$. The former can be easily verified. In fact, we only need to define $f_{x_0,\delta,\epsilon}$ as the same as $f(x)$ on $[x_0, x_0 + 2(\delta - \epsilon)]$, and as the line segment connecting the point $(x_0 + 2(\delta - \epsilon), f(x_0 + 2(\delta - \epsilon)))$ and $(x_0 + 2\delta, f(x_0))$, and then subtract from the resulting function its integral over $(0, 2\epsilon)$. As for the latter concern, it will become clear that in all cases the particular details of the modification are irrelevant.

This definition may appear to be rather complicated. However, as will be shown, its computation turns out to be quite easy.

From this definition we can draw the following conclusions.

Property 1. *For each $\theta \in (0, 2\pi)$, $D_\theta^\alpha f(x_0)$ is a local property of the function $f(x)$ in the vicinity of $x = x_0$.*

Proof. This is obvious because of the limiting process in the definition. □

Property 2. *If $f(x)$ is continuously differentiable at $x = x_0$, then $D_\theta^1 f(x_0) = f'(x_0+)$.*

Proof. Since on $[x_0, x_0+2(\epsilon-\delta)]$ $f_{x_0,\epsilon,\delta}(x)$ differs with $f(x)$ only by a constant, $f_{x_0,\epsilon,\delta}(x)$ must also have first order derivative on this interval and coincide with that of $f(x)$. Since f is continuously differentiable, as $x \to x_0$, $\frac{d}{dx} f_{x_0,\epsilon,\delta}(x) \to f'(x_0)$ □

Property 3. *If $|f(x) - f(x_0)| \le k\,|x - x_0|^\gamma$, then for $\alpha < \gamma$ we must have*

$$D_\theta^\alpha f(x_0) = 0. \tag{41}$$

Proof. If $f(x) \in H^\alpha$ and is 2π-periodic with zero mean, then for $\alpha < \gamma$ its Weyl derivative has the integral expression [11]

$$D^\alpha f(x) = \frac{1}{2\pi} \int_0^{2\pi} \{f(x-t) - f(x)\} \Psi'_{1-\alpha}(t)dt. \tag{42}$$

Therefore

$$D^\alpha f_{x_0,\delta,\epsilon}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \{f_{x_0,\delta,\epsilon}(\theta-t) - f_{x_0,\delta,\epsilon}(\theta)\} \Psi'_{1-\alpha}(t)dt. \tag{43}$$

Since $|f(x) - f(x_0)| \leq k\,|x-x_0|^\gamma$, by the property of $f_{x_0,\delta,\epsilon}$

$$|f_{x_0,\delta,\epsilon}(\theta-t) - f_{x_0,\delta,\epsilon}(\theta)| \leq K \left|\frac{\delta}{\pi}t\right|^\gamma. \tag{44}$$

Since (34)

$$|f_{x_0,\delta,\epsilon}(\theta-t) - f_{x_0,\delta,\epsilon}(\theta)|\,|\Psi_{1-\alpha}(t)| \leq \frac{KC_\alpha}{\pi^\gamma} |t|^{-1-\alpha+\gamma} \delta^\gamma, \tag{45}$$

which gives

$$\left|\left(\frac{\pi}{\delta}\right)^\alpha D^\alpha f_{x_0,\delta,\epsilon}(\theta)\right| \leq \frac{KC_\alpha}{2\pi^{1-\alpha+\gamma}} \delta^{\gamma-\alpha} \int_0^{2\pi} |t|^{-1-\alpha+\gamma}\, dt. \tag{46}$$

If $\alpha < \gamma$, the integral in the right hand side of the above equation converges. Due to the factor $\delta^{\gamma-\alpha}$,

$$\lim_{\epsilon\to 0} \left|\left(\frac{\pi}{\delta}\right)^\alpha D^\alpha f_{x_0,\delta,\epsilon}(\theta)\right| = 0. \tag{47}$$

Then it follows that the α^{th} local fractional derivative is zero

$$D_\theta x^\alpha f(x_0) = 0. \tag{48}$$

□

The next conclusion deals with the case of $\alpha = \gamma$.

Proposition 4. *Let $0 < \alpha < 1$ and $f \in H^\alpha$ in a neighborhood of x_0. Then the local fractional derivative of order α at x_0 is*

$$\begin{aligned} D_\theta^\alpha f(x_0) = \lim_{x\to 0} \frac{d}{dx} \int_0^x \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} (f(x_0+t) - f(x_0))dt \\ + A(\theta) \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h^\alpha} dt, \end{aligned} \tag{49}$$

where

$$A(\theta) = \int_0^{2\pi} r'_{1-\alpha}(\theta-t) t^\alpha dt. \tag{50}$$

Proof. The Weyl fractional integral of order $1-\alpha$ of the 2π-periodic function $f_{x_0,\delta,\epsilon}(x)$ is

$$\begin{aligned} I^{1-\alpha} f_{x_0,\delta,\epsilon}(x) &= \frac{1}{2\pi}\int_0^{2\pi} \Psi_{1-\alpha}(x-t) f_{x_0,\delta,\epsilon}(t)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\{ \frac{2\pi}{\Gamma(1-\alpha)}(x-t)_+^{-\alpha} + r_{1-\alpha}(x-t) \right\} f_{x_0,\delta,\epsilon}(t)dt \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha} f_{x_0,\delta,\epsilon}(t)dt + \frac{1}{2\pi}\int_0^{2\pi} r_{1-\alpha}(x-t) f_{x_0,\delta,\epsilon}(t)dt. \end{aligned} \tag{51}$$

Since for $0 \le x \le \pi$ and $0 < t < x$, $f_{x_0,\delta,\epsilon}(t) = f(\frac{\delta}{\pi}x + x_0) - c$

$$\begin{aligned} \int_0^x (x-t)^{-\alpha} f_{x_0,\epsilon}(t)dt &= \int_0^x (x-t)^{-\alpha} \left\{ f(\frac{\delta}{\pi}t + x_0) - c \right\} dt \\ &= \int_0^x (x-t)^{-\alpha} \left\{ f(\frac{\delta}{\pi}t + x_0) - f(x_0) \right\} dt + \frac{f(x_0)-c}{1-\alpha} x^{1-\alpha}. \end{aligned} \tag{52}$$

From the definition of $f_{x_0,\delta,\epsilon}$ we have

$$\begin{aligned} &\int_0^{2\pi} r_{1-\alpha}(x-t) f_{x_0,\delta,\epsilon}(t)dt = \int_0^{2\pi} r_{1-\alpha}(x-t)\left(f\left(\frac{\delta}{\pi}t + x_0\right) - c\right) dt \\ &= \int_0^{2\pi} r_{1-\alpha}(x-t)\left(f\left(x_0 + \frac{\delta}{\pi}t\right) - f(x_0) + \xi(t)\right) dt + (f(x_0) - c)\int_0^{2\pi} r_{1-\alpha}(x-t)dt, \end{aligned} \tag{53}$$

where

$$\xi(t) = f_{x_0,\delta,\epsilon}(t) - f\left(\frac{\delta}{\pi}t + x_0\right) + c. \tag{54}$$

From (32) we can find that $\int_0^{2\pi} \Psi_{1-\alpha}(x-t)dt = 0$, therefore

$$\begin{aligned} \int_0^{2\pi} r_{1-\alpha}(x-t)dt &= \int_0^{2\pi} \Psi_{1-\alpha}(x-t)dt - \int_0^x (x-t)^{-\alpha}dt \\ &= \int_0^{2\pi} \Psi_{1-\alpha}(x-t)dt - \frac{x^{1-\alpha}}{1-\alpha} = -\frac{x^{1-\alpha}}{1-\alpha}. \end{aligned} \tag{55}$$

By putting the above equations together,

$$\begin{aligned} \frac{d}{dx} I^{1-\alpha} f_{x_0,\delta,\epsilon}(x) =& \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha} \left\{ f(\frac{\delta}{\pi}t + x_0) - f(x_0) \right\} dt \\ &+ \frac{1}{2\pi}\frac{d}{dx}\int_0^{2\pi} r_{1-\alpha}(x-t)\left(f\left(x_0 + \frac{\delta}{\pi}t\right) - f(x_0)\right) dt \\ &+ \frac{1}{2\pi}\frac{d}{dx}\int_0^{2\pi} r_{1-\alpha}(x-t)\xi(t)dt. \end{aligned} \tag{56}$$

Now we evaluate the three terms of the right hand side of the above respectively as $\delta \to 0$.

$$\begin{aligned}
&\left(\frac{\pi}{\delta}\right)^{\alpha} \frac{d}{dx}\int_0^x (x-t)^{-\alpha}\left\{f(\frac{\delta}{\pi}t + x_0) - f(x_0)\right\} dt \\
&= \frac{\pi}{\delta}\frac{d}{dx}\int_0^{\frac{\delta}{\pi}x} (\frac{\delta}{\pi}x - t)^{-\alpha}\left\{f(t+x_0) - f(x_0)\right\} dt \\
&\to \lim_{x\to 0}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}\left\{f(t+x_0) - f(x_0)\right\} dt.
\end{aligned} \tag{57}$$

For the second term,

$$\begin{aligned}
&\left(\frac{\pi}{\delta}\right)^{\alpha} \frac{d}{dx}\int_0^{2\pi} r_{1-\alpha}(x-t)\left(f\left(x_0 + \frac{\delta}{\pi}t\right) - f(x_0)\right) dt \\
&= \frac{d}{dx}\int_0^{2\pi} r_{1-\alpha}(x-t)t^{\alpha}\left(f\left(x_0 + \frac{\delta}{\pi}t\right) - f(x_0)\right)\left(\frac{\pi}{\delta t}\right)^{-\alpha} dt \\
&\to A(x)\lim_{x\to x_0}\frac{f(x_0+h) - f(x_0)}{h^{\alpha}}.
\end{aligned} \tag{58}$$

Finally, since r is continuously differentiable in $(-2\pi, 2\pi]$, for $x \in (0, 2\pi)$, $t \in [0, 2\pi]$, $x - t \in [x - 2\pi, x]$ and therefore $r'(x-t)$ must be bounded

$$\begin{aligned}
\left|\left(\frac{\pi}{\delta}\right)^{\alpha} \frac{d}{dx}\int_0^{2\pi} r_{1-\alpha}(x-t)\xi(t)dt\right| &= \left|\left(\frac{\pi}{\delta}\right)^{\alpha}\int_0^{2\pi} r'_{1-\alpha}(x-t)\xi(t)dt\right| \\
&< B\frac{\epsilon}{\delta^{\alpha}} \to 0.
\end{aligned} \tag{59}$$

By putting all this together we have proved the conclusion. □

Note that the first term of (49) is the same as the K-G local fractional derivative. The second term is a multiple of the limit of $\frac{\Delta y}{\Delta x^{\alpha}}$. The two terms coincide up to a constant factor when $I^{1-\alpha}$ is continuously differentiable. Therefore we can see that the approach via Weyl fractional derivative reaches the same type of description of the local fractional smoothness of functions.

6 Conclusion

We analyzed the Kolwankar-Gangal definition of local fractional derivative and proposed a new approach that is more concise and easier to calculate. Functions may have a non-trivial local fractional derivative on a fractal set. The local differentiability is also connected with the characteristic equation of probability distributions related to stable distributions. Local fractional derivative can also be based on the Weyl derivative. The resulting local fractional derivative is in some sense equivalent to the one based on the Riemann-Liouville derivative.

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