FRACTIONAL DIFFERENTIAL EQUATIONS AND STABLE DISTRIBUTIONS

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Abstract

A partial differential equation of fractional order is introduced whose solution gives nearly all the stable distributions. From this equation a class of probability distributions is found whose infinite convolutions generate the stable distributions.

Keywords: Fractional derivatives, Fractional differential equations, stable distributions, stable densities, infinitesimal generator

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1. Introduction

Stable distributions generalize the normal distribution to random variables with infinite variance, infinite mean, or both. A classical review of this field is [8], however the reader should be warned that the modern notation in this field is somewhat different. More modern summaries are provided in [9, 27, 19]. Because of their generality, stable distributions are of considerable interest to stochastic modellers. In this regard they have been employed in several fields. Since [16] they have been used to model financial stock returns, see also [17] and the references therein. In astrophysics [3], the Holtsmark distribution describes the gravitational force at a given point of all the stars in the universe. Stable distributions have also been used in vortex models of turbulence [28].

However, apart from some special cases, only the characteristic functions (Fourier transforms) of stable density functions are explicitly known. This makes it difficult to build intuition about a given stable distribution.

In recent years fractional calculus [25, 21, 26] has received much interest due to its new applications. In physics a short but by no means exhaustive list of references

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includes [4, 12, 13, 18, 11, 30]. A recent set of applications in mechanical engineering may be found in [1], while an electrical engineering control theory approach may be found in [15]. In quantitative finance the work of [24] is particularly interesting. In econometrics fractional signal processing is described in [2, p. 60].

Perhaps surprisingly, partial differential equations (PDE) of fractional order have been linked with stable distributions. At this point it makes sense to note that there are two kinds of Lévy distributions – the Lévy stable distributions we discuss here, and another set (associated with the idea of Lévy flights; see also [4]) which incorporate power law tails. Fractional calculus was applied in [20] to generate distributions of this second, power-law tailed type. Although this paper is devoted to the links between fractional calculus and the Lévy stable distributions, we note that [20] helped inspire one of the authors of this paper to begin looking in this direction. The work of [10, 22] links the fractional Laplace operator to a symmetric subclass of Lévy stable distributions [29]. The contribution of this work is to generalize and extend these papers. We present a new PDE which uses the Riemann-Liouville [25, 6] fractional derivatives. This equation can generate not just the symmetric but nearly all the α-stable distributions. It has the defect that it cannot generate the \( \alpha = 1 \) distributions.

From this equation we also obtain a new class of probability density function which we call the generating densities of the stable distributions – the infinite convolution of the generating densities gives the densities of the stable distributions. These generating densities make it trivial to determine that the stable distributions are always positive, a fact which is often somewhat difficult to prove. We may also gain some insights into the properties of the stable distributions through examination of these generating densities.

These results are interesting because they connect two seemingly different fields. They allow important modelling intuition about stable distributions to be developed, and they perhaps also give some insight into one of the “holy grail” questions of fractional calculus, namely what fractional differentiation really “means”.

2. Some notation and standard definitions

In this paper we consider functions \( f(x, t) : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{C} \) which, when considered as functions of \( x \) alone indexed by \( t \) are \( L^1 \). The definition of the Fourier transform pair
we use is:

\[ F(\omega) = \mathcal{F}\{f\} = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx, \]
\[ f(x) = \mathcal{F}^{-1}\{F\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x)e^{i\omega x} d\omega. \] (1)

3. Stable Distributions

Stable distributions are generalizations of the Gaussian probability distributions which retain some of their important properties. Among these characteristics, the most important are stability and the domain of attraction.

A random variable \( X \) is said to have a stable distribution if for any positive numbers \( A \) and \( B \), there are real numbers \( C, D, \) and \( \alpha \), \( 2 > 0, \alpha \in (0, 2) \) such that \( C^\alpha = A^\alpha + B^\alpha \) and

\[ AX_1 + BX_2 = CX + D, \quad \text{in distribution,} \] (2)

where \( X_1 \) and \( X_2 \) are independent copies of \( X \). The Gaussian distribution is a special case of the stable distribution for which \( \alpha = 2 \).

Explicit expressions for stable densities are not in general known. However, the characteristic functions (Fourier transforms of densities) are known and provide an equivalent definition for stable distributions [27].

Given four parameters \( \alpha, \sigma, \beta, \) and \( \mu \), a stable distribution \( X = S_\alpha(\sigma, \beta, \mu) \) can be specified as follows. If the probability density function (PDF) of \( X \) is denoted by \( p_X(t) \), then the characteristic function of \( X \), or the Fourier transform of \( p_X \) is defined by

\[ \mathcal{F}\{p_X(\cdot)\} = \begin{cases} 
\exp\left\{ -\sigma^\alpha |\omega|^\alpha(1 + i\beta \text{sign}(\omega) \tan \frac{\alpha \pi}{2} + i\mu \omega) \right\}, & \alpha \neq 1 \\
\exp\left\{ -\sigma |\omega|(1 + i\beta \frac{2}{\pi} \text{sign}(\omega) \ln \omega) + i\mu \omega \right\}, & \alpha = 1,
\end{cases} \] (3)

where \( 0 < \alpha \leq 2, -1 \leq \beta \leq 1, \sigma > 0 \) and \(-\infty < \mu < \infty\).

The absolute value of the parameter \( \beta \in (-1, 1) \) controls the extent of asymmetry of the PDF. Therefore the case \( \beta = \pm 1 \) is called “the stable distribution of maximal skewness”. When \( \alpha = 2, \beta \) is irrelevant, and

\[ S_2(\sigma, 0, \mu) = N(\mu, 2\sigma^2) \] (4)

is the Gaussian distribution.
4. Fractional Derivatives

The idea of a fractional derivative is implicit in the notation $\frac{d^n}{dx^n}$ for the $n$th derivative, and appears to go back to Leibnitz [25] who developed this notation. There are many ways to interpolate between derivatives of integer order, [26] etc. each with its own benefits and disadvantages. One way involves the observation that the Fourier transform of $f^{(n)}(x)$ is $(i\omega)^n F(\omega)$, which might suggest the defining of the $\alpha$th derivative as the inverse Fourier transform of $(i\omega)^\alpha F(\omega)$.

Consider a function (see Section 2 for details) of two variables $f(x, t)$ from $-\infty < x < \infty, t \geq 0$ to the complex numbers. We define its fractional derivatives in $x$ based on the Fourier transform (1).

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x, t) = \mathcal{F}^{-1} \{ (i\omega)^\alpha \mathcal{F} \{ f \} \}$$  \hspace{1cm} (5)

$$\frac{\partial^\alpha}{\partial (-x)^\alpha} f(x, t) = \mathcal{F}^{-1} \{ (-i\omega)^\alpha \mathcal{F} \{ f \} \}$$  \hspace{1cm} (6)

where $\mathcal{F} \{ f \}$ is the Fourier transform with respect to $x$. Roughly speaking (5) is equivalent to the Riemann-Liouville fractional derivative [26]

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x, t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{x} \frac{f(y, t)dy}{(x-y)^{\alpha-n+1}},$$  \hspace{1cm} (7)

where $n = 1 + [\alpha]$, $[\alpha]$ is the largest integer not greater than $\alpha$.

One may wonder why we should have two definitions of the fractional order derivative: $\frac{\partial^\alpha}{\partial x^\alpha}$ and $\frac{\partial^\alpha}{\partial (-x)^\alpha}$, while we have only one for normal derivatives. In fact, in the case of integral order derivative we have a simple relation

$$\frac{\partial^n}{\partial (-x)^n} = (-1)^n \frac{\partial^n}{\partial x^n},$$  \hspace{1cm} (8)

so that we do not consider $\frac{\partial^\alpha}{\partial (-x)^\alpha}$ to be a distinct derivative. However, in the case of fractional order, there is no such simple relation like (8) linking (5) and (6). In this case we must retain both derivatives to make things complete.

The link between stable distributions and fractional differential equations has been considered in e.g. [10, 22, 29]. These authors define the fractional derivative via

$$\frac{\partial^\alpha}{\partial |x|^\alpha} f(x, t) = \mathcal{F}^{-1} \{ -|\omega|^\alpha \mathcal{F} \{ f \} \},$$  \hspace{1cm} (9)

which is called by some authors the symmetric fractional derivative.

It is easy to see that only when $\alpha = 2$ but not $\alpha = 1$ does (9) reduce to the regular integral order derivative. It deviates from the classical Riemann-Liouville-like
fractional derivative, and is actually a generalization of the Laplace operator
\[
\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2},
\]
which is the second derivative in the 1-dimensional case.

5. Fractional Partial Differential Equations

For parameters \(0 < \alpha \leq 2\), \(\alpha \neq 1\), \(-1 \leq \beta \leq 1\) and \(-\infty < \mu < \infty\), let
\[
c = \cos \frac{\alpha \pi}{2}, \quad s = \sin \frac{\alpha \pi}{2}.
\]
Define a fractional PDE by
\[
\frac{\partial u}{\partial t} = -\frac{1 + \beta}{2c} \frac{\partial^\alpha}{\partial x^\alpha} u(x, t) - \frac{1 - \beta}{2c} \frac{\partial^\alpha}{\partial (-x)^\alpha} u(x, t) + \mu \frac{\partial}{\partial x} u(x, t).
\]
Denote by \(H(\omega, t)\) the Fourier transform of \(u(x, t)\) with respect to \(x\). Definitions (5) and (6) yield
\[
\frac{\partial H}{\partial t} = -\frac{1 + \beta}{2c} (i\omega)^\alpha H - \frac{1 - \beta}{2c} (-i\omega)^\alpha H + (i\mu \omega) H,
\]
If the initial value \(u(x, t) = \delta(x)\), then \(H(\omega, 0) = 1\), and the solution is
\[
H(\omega, t) = \exp \left\{-\frac{1 + \beta}{2c} (i\omega)^\alpha t - \frac{1 - \beta}{2c} (-i\omega)^\alpha t + i\mu \omega t\right\}. \tag{14}
\]
For \(\alpha \in (0, 2]\),
\[
(i\omega)^\alpha = |\omega|^\alpha e^{i \text{sign}(\omega) \frac{\alpha \pi}{2}} = |\omega|^\alpha [c + i \text{sign}(\omega) s] \tag{15}
\]
\[
(-i\omega)^\alpha = |\omega|^\alpha e^{-i \text{sign}(\omega) \frac{\alpha \pi}{2}} = |\omega|^\alpha [c - i \text{sign}(\omega) s].
\]
Now from (14) and (15) the Fourier transform of the fundamental solution of (17) can be written as
\[
H(\omega, t) = \exp \left\{-\frac{1 + \beta}{2c} (i\omega)^\alpha t - \frac{1 - \beta}{2c} (-i\omega)^\alpha t + i\mu \omega t\right\}
\]
\[
= \exp \left\{-\frac{1 + \beta}{2c} |\omega|^\alpha t [c + i \text{sign}(\omega) s] - \frac{1 - \beta}{2c} |\omega|^\alpha t [c - i \text{sign}(\omega) s] + i\mu \omega t\right\}
\]
\[
= \exp \left\{-|\omega|^\alpha t - i\beta \text{sign}(\omega) \tan \frac{\alpha \pi}{2} |\omega|^\alpha t + i\mu \omega t\right\}. \tag{16}
\]
By comparison to (3) this proves that, provided \(\alpha \neq 1\), \(u(x, t)\) considered a function of \(x\) is the probability density function of the stable distribution \(S_\alpha(t^{1/\alpha}, \beta, \mu)\). This
provides a direct connection between stable distributions and a class of fractional partial differential equations.

We note that another fractional PDE is defined in [14]

\[
\frac{\partial u}{\partial t} = -\beta \frac{\partial^\alpha}{c \partial x^\alpha} u(x, t) + (1 - \beta) \frac{\partial^\alpha}{\partial |x|^\alpha} u(x, t) + \mu \frac{\partial}{\partial x} u(x, t),
\]

(17)

\[u(x, 0) = u_0(x).
\]

(18)

where \(-\infty < x < \infty\) and \(t > 0\). It is easily verified that this equation is equivalent to (12).

6. The Generating Densities

The densities of the stable distributions are generally not known. Instead, the characteristic functions are used to identify the stable distributions. Here through the fractional PDE we define the generating densities for probability distributions and calculate them. The generating distributions are easier to handle and can help characterize the stable densities.

Since the parameter \(\mu\) in the fractional PDE (12) represents a simple translation, we henceforth set \(\mu = 0\). In this section of the paper we will need to use some machinery of classical functional analysis. The reader who is both unfamiliar with these tools and happy to take these results on faith is invited to skip to the last few paragraphs of this section and to read section 6.1 for some intuitive motivation.

**Definition 1.** For \(0 < \alpha < 2\), \(\alpha \neq 1\) and \(D > 0\), define

\[
\mathbb{D} = \{ u \in L^1(\mathbb{R}) : \mathcal{F}^{-1}(\{ (i\omega)^\alpha \mathcal{F}\{u\}\}) , \mathcal{F}^{-1}(\{ (-i\omega)^\alpha \mathcal{F}\{u\}\}) \in L^1(\mathbb{R}) \}.
\]

(19)

For \(u \in \mathbb{D}\) define the fractional differential operator \(\mathcal{A}\) by

\[
\mathcal{A}u = \mathcal{F}^{-1}\left\{ -\left[ \frac{1 + \beta}{2c}(i\omega)^\alpha + \frac{1 - \beta}{2c}(-i\omega)^\alpha \right] \mathcal{F}\{u\} \right\},
\]

(20)

**Theorem 1.** For \(\lambda > 0\), there exists a probability density function \(g_\lambda(x)\), \(-\infty < x < \infty\), such that

\[
\lambda^{-1}(\lambda^{-1} - \mathcal{A})^{-1} u = g_\lambda * u.
\]

(21)

**Proof.** From (20) we obtain the Fourier transform

\[
\mathcal{F}\{(\lambda - \mathcal{A})^{-1} u\} = \left[ \lambda^{-1} + \frac{1 + \beta}{2c}(i\omega)^\alpha + \frac{1 - \beta}{2c}(-i\omega)^\alpha \right]^{-1} \mathcal{F}\{u\}.
\]

(22)
If we let
\[ G(\omega, \lambda) = \lambda^{-1} \left[ \lambda^{-1} + \frac{1 + \beta}{2c}(i\omega)^\alpha + \frac{1 - \beta}{2c}(-i\omega)^\alpha \right]^{-1}, \] (23)

then
\[ g_\lambda(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega, \lambda)e^{i\omega x} d\omega, \quad G(\omega, \lambda) = \int_{-\infty}^{\infty} g_\lambda(x)e^{-i\omega x} dx. \] (24)

Therefore \[ \int_{-\infty}^{\infty} g_\lambda(x) dx = 1. \] In the next section we calculate \( g_\lambda(x) \) and show that \( g_\lambda(x) \geq 0 \), which will complete the proof.

This conclusion shows that \( \lambda \mathcal{R}(\lambda, \mathcal{A}) \) is \( \mathcal{A} \) is the infinitesimal generator of a strongly continuous semigroup \( \mathcal{T}(t) \) of operators in \( L^1(\mathbb{R}) \). Since the resolvent operator is positive so is \( \mathcal{T}(t) \). It is now easy to show that \( \|\mathcal{T}(t)u\| = \|u\| \), i.e., \( \mathcal{T}(t) \) is a stochastic semigroup. This justifies the formal calculations of the last section. Hence we know that the semigroup \( \mathcal{T}(t) \) can be written as
\[ \mathcal{T}(t)u = h_t * u \] (25)
where \( h_t(x) \) is the density of the stable distribution
\[ S_{\alpha}(t^{1/\alpha}, \beta, 0) \] (26)

Since we have seen that \( \mathcal{T}(t) \) involves the stable distributions, it would be of interest to determine the relationship between \( g_\lambda(x) \) and the stable distributions. In fact, we can prove that
\[ h_t = \lim_{n \to \infty} g_{t/n} * g_{t/n} * \ldots * g_{t/n}. \] (27)

Because of this, we call \( g_\lambda(x) \) the generating density of the stable distribution (26).

Here is a simple proof of (27). Denote by \( \Omega \) the expression
\[ \Omega = \frac{1 + \beta}{2c}(i\omega)^\alpha + \frac{1 - \beta}{2c}(-i\omega)^\alpha. \] (28)

From (14) we see that, for \( \mu = 0 \), the Fourier transform of \( h_t \) is
\[ \mathcal{F}\{h_t\} = H(\omega, t) = e^{-i\Omega t}, \] (29)
while from (23) the Fourier transform of \( g_\lambda \) is
\[ \mathcal{F}\{g_\lambda\} = G(\omega, \lambda) = (1 + \lambda\Omega)^{-1} \] (30)
Using the convolution theorem for Fourier transforms and the limiting definition for exponential functions from basic calculus yields the result

\[ H(\omega, t) = \lim_{n \to \infty} [G(\omega, t/n)]^n. \tag{31} \]

Another application of the convolution theorem leads to (27).

The fact that for any \( \lambda > 0 \), \( G(\omega, \lambda) \) is the characteristic function also easily proves that \( H(\omega, t) \) is the characteristic function of an infinitely divisible distribution [8], i.e., for any integer \( n > 0 \), \( H(\omega, \lambda) \) can be written as \( h^n(\omega, \lambda) \) where \( h(\omega, \lambda) \) is a characteristic function. In fact

\[ H(\omega, t) = e^{-\Omega t} = e^{-n(1-\Omega t/n - 1)} = \lim_{n \to \infty} e^{-n(1+\Omega t/n - 1)} = \lim_{n \to \infty} e^{-n(G(\omega, t/n))} \tag{32} \]

and then DeFinetti’s theorem [23, p. 19] can be used to reach the conclusion.

6.1. Generating Densities as Intuition Builders

What intuition may be we build by considering generating densities? We answer that question with a question. Which gives more intuition about geometric growth of a population \( N(t) \), the statement \( N(t) = N_0 \exp(\mu t) \) or the differential equation \( N(t) = \mu N(t), N(0) = N_0 \)? The two expressions complement one another. The first tells us that growth is exponential. However note that this is a valuable insight only because we know so much about the exponential function and its properties. The second tells us that the growth at a given instant is proportional to the population at that instant. If we look at a short instant \( \Delta t = t/n \) over which population is relatively constant, we find that \( g(t + \Delta t) = (1 + \mu \Delta t)g(t) \). Putting this together yields

\[ N(t) = \lim_{n \to \infty} (1 + \frac{t}{n})^n. \]

When we turn to densities we can think of the generating density as being the Green’s function which, convolved with the density at time \( t \), returns the density at time \( t + \Delta t \). So, in the same way as in the above example, the generating density says something about how to extend a solution over a short time interval. In the next subsection we will develop some formulae for the generating density of a normal distribution and show what insights these expressions yield.
6.2. The Generating Density of the Gaussian distribution

To build intuition, we present the generating density for the Gaussian distribution. First note that when we choose $\alpha = 2$ and $\mu = 0$ in (11) and (13) we obtain the diffusion equation of mathematical physics. The solutions of the IVP of the diffusion equation $u_t = u_{xx}$ form a semigroup. For each $t > 0$ the solution

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t},$$

is a PDF. By combining (11) and (28) we see that the generating characteristic function is

$$G(\omega, \lambda) = \frac{1}{1 + \lambda\omega^2}.$$  

(34)

We can compute the inverse transform of this using (1) using the residue theorem after making the change of variables $z = i\omega$. Note that the resulting integral is along the whole imaginary axis, and the integrand has two simple poles symmetrically located on the real axis. By picking a contour which will vanish in the infinite limit, we see that we must integrate around one of these poles when $x > 0$ and around the other pole when $x < 0$. With this it is easy to see that the generating density is

$$g_\lambda(x) = \begin{cases} \frac{1}{2\sqrt{\lambda}} e^{x^2/\lambda}, & x < 0 \\ \frac{1}{2\sqrt{\lambda}} e^{-x^2/\lambda}, & x > 0. \end{cases}$$

(35)

Now what intuition does this bring? We recall that this expression tells us with what we must convolve the solution of the diffusion equation at time $t$ ($u(x, t)$) to obtain its solution at time $t + \lambda$, provided that $\lambda$ is small. The linearity of the heat equation allows us, without loss of generality, to suppose that we are starting from a delta function $u(x, t) = \delta(x - x_0)$. Let’s change perspective now and suppose that we are simulating the solution of the diffusion equation. Our simulation has taken us to $x = x_0$ at time $t$. To where will it lead us at time $t + \Delta t$? To do this we can write (35) as

$$g_{\Delta t}(x) = \frac{1}{2\sqrt{\Delta t}} \exp\left(-\sqrt{\frac{x^2}{\Delta t}}\right).$$

Then we can conclude that probability that a particle at location $x_0$ at time $t$ lies outside the interval $(x_0 - \frac{\Delta x}{\sqrt{\Delta t}}, x_0 + \frac{\Delta x}{\sqrt{\Delta t}})$ is

$$\int_{\frac{\Delta x}{\sqrt{\Delta t}}}^{\infty} \exp\left(-\frac{x}{\sqrt{\Delta t}}\right) \frac{dx}{\sqrt{\Delta t}} = \int_{\Delta x}^{\infty} \exp(-u)du = \exp(-\Delta x).$$

(36)
This convinces us of the standard result that the step size for simulating the diffusion equation by means of a random walk must scale like the square root of the time step. From this also follows the well known fact that the speed limit of diffusion is infinite.

In the next section we continue with more analytic results about the generating densities of general stable distributions.

7. Calculation of the Generating Densities

In this section we calculate the function $g_\lambda(x)$

$$g_\lambda(x) = \frac{1}{2\pi i \lambda} \int_{-\infty}^{\infty} \left[ \lambda^{-1} + \frac{1+\beta}{2c} (i\omega)^\alpha + \frac{1-\beta}{2c} (-i\omega)^\alpha \right]^{-1} e^{i\omega x} d\omega. \quad (37)$$

and show $g_\lambda(x) \geq 0$, and complete the proof of Theorem 1.

By setting $z = i\omega$, and

$$f(z) = \lambda^{-1} + \frac{1+\beta}{2c} z^\alpha + \frac{1-\beta}{2c} (-z)^\alpha. \quad (38)$$

we have

$$g_\lambda(x) = \frac{1}{2\pi i \lambda} \int_{-i\infty}^{i\infty} \frac{e^{ixz}}{f(z)} dz. \quad (39)$$

For, $z = re^{i\theta}$, $-\pi < \theta \leq \pi$, $z^\alpha$ and $(-z)^\alpha$ are defined by

$$z^\alpha = r^\alpha e^{i\alpha\theta}, \quad \text{and} \quad (-z)^\alpha = \begin{cases} r^\alpha e^{i\alpha(\theta-\pi)}, & 0 < \theta \leq \pi \\ r^\alpha e^{i\alpha(\theta+\pi)}, & -\pi < \theta < 0 \end{cases} \quad (40)$$

respectively. Hence, the function $f(z)$ is single-valued and analytic on the complex plane except on the following set

$$\begin{align*}
\{ & \theta = 0, \theta = \pi, \quad -1 < \beta < 1 \\
& \theta = \pi, \quad \beta = 1 \\
& \theta = 0, \quad \beta = -1 
\end{align*} \quad (41)$$

Only when $\alpha > 1$, $\beta = 1$ or $\beta = -1$ does $f(z)$ have zeros, which are

$$P = \left( -\frac{c}{\lambda} \right)^{1/\alpha}, \quad \alpha > 1, \beta = 1$$

$$Q = -\left( -\frac{c}{\lambda} \right)^{1/\alpha}, \quad \alpha > 1, \beta = -1 \quad (42)$$

The following is obvious for $\alpha > 1$ and can be proved by Jordan’s Lemma [5, p. 175] when $\alpha < 1$

$$\lim_{R \to \infty} \int_{|z|=R, \ \Re z > 0} \frac{e^{-z|x|}}{f(z)} dz = 0. \quad (43)$$
Based on these we have

$$
g_{\lambda}(x) = \begin{cases} 
\frac{1}{2\pi\lambda i} \int_{C(\lambda, 0)} e^{\lambda z} f(z) \, dz, & x < 0 \\
\frac{1}{2\pi\lambda i} \int_{C(-\lambda, 0)} e^{\lambda z} f(z) \, dz, & x > 0 \\
\frac{1}{\lambda} \text{Res} \left\{ \frac{e^{\lambda z}}{f(z)}, z = P \right\}, & x < 0 \\
\frac{1}{2\pi\lambda i} \int_{C(\lambda, 0)} e^{\lambda z} f(z) \, dz, & x > 0 \\
\frac{1}{\lambda} \text{Res} \left\{ \frac{e^{\lambda z}}{f(z)}, z = Q \right\}, & x > 0 \\
0 & x < 0 \\
\frac{1}{2\pi\lambda i} \int_{C(-\lambda, 0)} e^{\lambda z} f(z) \, dz, & x > 0 \\
\frac{1}{2\pi\lambda i} \int_{C(\lambda, 0)} e^{\lambda z} f(z) \, dz, & x < 0 \\
0 & x > 0 
\end{cases} \quad (44)$$
Note that
\begin{align*}
(-z)^\alpha &= z^\alpha (\cos \alpha \pi + i \sin \alpha \pi), \quad z \in \overline{OA} \\
(-z)^\alpha &= z^\alpha (\cos \alpha \pi - i \sin \alpha \pi), \quad z \in \overline{AO} \\
z^\alpha &= (-z)^\alpha (\cos \alpha \pi + i \sin \alpha \pi), \quad z \in \overline{OB} \\
z^\alpha &= (-z)^\alpha (\cos \alpha \pi - i \sin \alpha \pi), \quad z \in \overline{BO}
\end{align*}

The result is

\begin{align*}
g_\lambda(x) &= \begin{cases} 
\frac{(1 - \beta) s}{\lambda \pi} \int_0^\infty \frac{z^\alpha e^{-xz} dz}{\alpha \lambda (-c/\lambda)^{1-1/\alpha}} & x < 0 \\
\frac{(1 + \beta) s}{\lambda \pi} \int_0^\infty \frac{z^\alpha e^{-xz} dz}{(\lambda^{-1} + (c(1 - \beta) + \beta c^{-1} z^\alpha)^2 + (1 - \beta) s^2 z^{2\alpha}} & x > 0 \\
\frac{2 s}{\lambda \pi} \int_0^\infty \frac{z^\alpha e^{-xz} dz}{(\lambda^{-1} + (c(1 + \beta) - \beta c^{-1} z^\alpha)^2 + (1 + \beta) s^2 z^{2\alpha}} & x < 0 \\
\frac{2 s}{\lambda \pi} \int_0^\infty \frac{z^\alpha e^{-xz} dz}{(\lambda^{-1} + (2c - c^{-1}) z^\alpha)^2 + 4 s^2 z^{2\alpha}} & x > 0 \\
\frac{\alpha \lambda (-c/\lambda)^{1-1/\alpha}}{e^{-(-c/\lambda)^{1/n} x}} & x < 0 \\
\frac{\alpha \lambda (-c/\lambda)^{1-1/\alpha}}{e^{-(-c/\lambda)^{1/n} x}} & x > 0 \\
0 & \beta = 1, \alpha > 1 \\
0 & \beta = -1, \alpha > 1 \\
0 & \beta = 1, \alpha < 1 \\
0 & \beta = -1, \alpha < 1 \\
0 & \beta = 1, \alpha > 1 \\
0 & \beta = -1, \alpha > 1 \\
0 & \beta = 1, \alpha < 1 \\
0 & \beta = -1, \alpha < 1
\end{cases}
\end{align*}

We can see that the integrals in (46) converge, and for all \( x \), \( g_\lambda(x) \geq 0 \), which proves that each \( g_\lambda \) is a PDF.

It is easily seen that as \( \beta = 0 \), \( g_\lambda(x) \) is an even function

\begin{align*}
g_\lambda(x) &= \frac{s}{\lambda \pi} \int_0^\infty \frac{z^\alpha e^{-|z|x} dz}{(\lambda^{-1} + c z^\alpha)^2 + s^2 z^{2\alpha}} 
\end{align*}

The results for \( \beta = \pm 1 \) give strong intuition about the behavior of these two strongly skewed distributions.
8. Conclusion

We defined a simple fractional differential equation whose solution leads to most of the stable distributions. The exception is that corresponding to the parameter choice $\alpha = 1$. A by-product of studying this fractional PDE with the operator semigroup theory is the discovery of a simple new approach to justifying that the functions (3) used to define stable distributions are indeed characteristic functions of probability distributions. Through this we also found a class of probability density functions which we call generating densities of stable distributions since the infinite convolutions of them lead to asymptotic expressions of the stable densities. These densities turn out to be simpler than the stable densities, and certain special cases have explicit expressions. Examination of these generating densities helps build intuition about the properties of stable distributions.

References


