Approximate Recursive Valuation of Electricity Swing Options

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Abstract

Pricing options on electrical power is important because of the worldwide trend toward deregulated electricity markets. It is fascinating because of the exotic nature of most power derivatives and because of the many unique features of electricity markets. One of these unique features is that electrical power cannot, in appreciable quantities, be stored. While this plays havoc with arbitrage-free pricing strategies, non-storability and the near inelasticity of electricity demand over short times make possible the supply-demand modelling of electricity spot prices. We use these insights to create an approximate pricing model for electricity spot prices and use this model to approximate the early exercise boundary for simple swing options.

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1 Introduction

A financial option is a security which, in exchange for initial payment, gives its holder the right, but not the obligation, to make a financial transaction at preset terms within some future time period. Options are used both to speculate on future price changes and to hedge risks arising from uncertain future prices. For general information see [1]. For application to an electricity market setting see [2] and [3].

Pricing an option requires two main ingredients: a (typically stochastic) price model for the underlying asset and a means for pricing risk. In what follows we concentrate on the first of these requirements and ignore the second. Our justification for this is as follows:

The electrical power market is radically incomplete. As a result, options on electricity cannot be priced using the usual replicating no-arbitrage strategies. We choose to price options using a “present value of the expected value” methodology operating in a Monte Carlo framework. Of course, in an incomplete market, an immediate question is how to discount for risk. This question has stumped the best financial minds throughout history and we do not presume to answer it here. Instead we follow engineering practice and choose an empirically selected discount factor greater than or equal to the risk-free rate.

Now we turn our attention to our price model. Our price model, described in [4], is a hybrid switching model which combines stack-based pricing ideas with the probabilistic approach which works so well with options on “paper” financial contracts. Our switching model is based on the observation that electricity price time series contain occasional large price “spikes”. Although these spikes are infrequent, they have a large financial impact, and risk management tools cannot ignore them. Our model works by understanding the partially deterministic and partially stochastic dynamics of available load $C(t)$ and electricity demand $D(t)$. The ratio of these is then computed to get the “lamination” $\alpha(t) = \frac{D(t)}{C(t)}$. This is used to find the spike probability $\epsilon(t) = F(\alpha(t))$ where $F$ is a function with an abrupt threshold at about $\alpha = 0.85$. Then in each time interval there is a spike with (time varying) probability $\epsilon(t)$ and no spike with probability $1 - \epsilon(t)$. If there is a spike the price is drawn from a “high” distribution, if not, the price is drawn from a “low” distribution. In [4] we chose a normal distribution for the “high” and a lognormal distribution for the “low” prices.
This model also requires specification of the demand and availability processes. Demand fluctuates about a seasonal mean and is largely temperature driven. A decent model ([5],[6]) of the daily temperature is an Ornstein-Uhlenbeck process with seasonally dependent volatility which reverts to a seasonal average temperature.

Supply is governed by plant outages, both planned and unplanned. Construction of modules which describe these phenomenon in good but not overwhelming detail is currently underway [7].

There is another difficulty inherent in using our model for pricing options with an early exercise feature. In this report we shall consider the problem of pricing a class of swing options, but the problem exists even for pricing the relatively simple Bermudan options.

The problem is: when early exercise is possible, pricing an option requires an optimal exercise strategy. But determining this optimal exercise strategy requires a sequence of option prices. Pricing the resulting large set of options using Monte Carlo methods is computationally intractable for all but the simplest situations. It is for this reason that Monte Carlo was long thought to be impractical for pricing American-style options. In fact some recent discoveries by Tilley [8], Broadie & Glasserman [9] and others (see review in [10]) have shown this to be untrue. We shall use one of the insights of these modern Monte Carlo pricing experts, together with some special features of our \( \alpha \)-model for generating spot electricity prices, to obtain an algorithm for rapidly determining a near-optimal exercise strategy for a swing option on electrical power. With this exercise strategy in hand a Monte Carlo pricing strategy again becomes feasible. The resulting Monte Carlo prices will be an underestimate for the true options price, obtained with the optimal exercise strategy. Our prices will, however, be useful not only for building insight but also as a starting point for more sophisticated numerical pricing algorithms.

This paper presents only the idea behind this framework, using some very simplified assumptions. We will price a swing with a high strike price on weekly average on-peak electricity prices. Our model will include diurnal variations in \( \epsilon(t) \) and will model heat waves and power plant failure by a simple Markov chain on daily time scales.

The basic idea is to recast the alpha model into a form in which, on a very short time scale, \( \alpha(t) \) and hence \( \epsilon(t) \) are driven by a Markov process. We then show that the resulting price process is given by a mixture of Poisson distributions, the details of which depend on the Markov process we choose. If we further assume that the weekends which punctuate the on-peak price
time series are enough to make the prices in one week at least approximately independent of the prices in the preceding week, we are able to consider average on-peak weekly (and hence also average on-peak monthly) prices as independent draws from a relatively simple distribution. This assumption suffices to make the problem of determining an approximate early exercise boundary tractable.

The paper is divided into three more sections. In Section 2 we discuss a mixture of Poissons reformulation of the hybrid model of [4]. In Section 3 we discuss swing options and present our method for approximating the exercise boundary for a weekly swing within this framework. We include an example of how this method would work with a very simple model here. We conclude in Section 4.

2 A Mixture of Poissons Reformulation

Our switching model works by understanding the partially deterministic and partially stochastic dynamics of available load $C(t)$ and electricity demand $D(t)$. The ratio of these is then computed to get a “lamination” parameter $\alpha(t) = \frac{D(t)}{C(t)}$, and finding the spike probability $\epsilon(t) = F(\alpha(t))$ where $F$ is a function with an abrupt threshold at about $\alpha = 0.85$. Then in each time interval there is a spike with (time varying) probability $\epsilon(t)$ and no spike with probability $1 - \epsilon(t)$.

This model contains a very definite time scale. For instance, if we model hourly prices this way, and if $\epsilon$ is constant, prices from a 16-hour on-peak day are drawn from a binomial distribution with 17 possible states (zero spikes, one spike, two spikes, ... sixteen spikes). If $\epsilon$ is also very small, only the one- and zero- spike cases are very likely to happen, and we recover a switching variable between the same high and low prices as before but with a spike probability 16 times as great. However, the idea that there is no serial correlation between price spikes within a day is not very reasonable.

This motivates the somewhat counterintuitive fine-graining we are about to do.

Suppose that our underlying probability model is for the price of the delivery of a megaWatt of power for a time interval $dt$ (measured in hours). The price for the power is most likely to be very small (free) but in an extremely unlikely case with probability $\epsilon dt$ it is actually $1000$ per megaWatt $dt$. Of course this model can only make any sense if $dt$ is very small and if
prices for successive time intervals are at worst weakly correlated, or it would end up with nontrivial probabilities of electricity prices of millions of dollars per MW-h. In this idealized model we assume that prices of nearby instants are drawn completely independently of one another. However, the probability with which spikes occur can incorporate some serial correlation.

Now let’s work out the power price for an hour made up of \( N \) instants each of length \( dt \). If there are \( k \) spikes in the hour the cost of a MW-h is \( k \times \$1000 \). The probability that there are \( k \) spikes in the hour is, of course, a binomial random variable with

\[
P(X = k\$1000) = \frac{N!}{(N-k)!k!} (\epsilon dt)^k (1 - \epsilon dt)^{N-k}.
\]  
(2.1)

Now assume that \( \epsilon N dt = \epsilon \ll 1 \). In this limit the probability of no price spikes, or \( X = 0 \), is approximately \( 1 - N \epsilon dt = 1 - \epsilon \), while the probability of a single price spike, or \( X = \$1000 \), is approximately \( N \epsilon dt = \epsilon \), and we recover, at least approximately, our original switching model.

We take the limit of the Binomial distribution as \( N \to \infty \) and \( dt \to 0 \) so that \( N dt = t \). The resulting price distribution (Section 2.4 of [11]) is the Poisson distribution

\[
P(X = k\$1000) = \frac{(\epsilon t)^k}{k!} \exp(-\epsilon t).
\]  
(2.2)

This gives us a new price model linked to the old one. This means that we can still estimate the relevant parameters from fundamental engineering data.

However, as we shall see next, this Poisson framework is also a natural one for including deterministic time-dependent \( \epsilon(t) \) as well as \( \epsilon(t) \) which fluctuates according to a Markov process.

### 2.1 Benefits of the Poisson Model

There are several benefits associated with the Poisson formulation when time-dependent switching parameters \( \epsilon(t) \) are introduced, both when \( \epsilon(t) \) is deterministic and when it is Markov driven.

#### 2.1.1 Time-Varying, Deterministic Switching Probability

Of course the benefits of a Poisson model begin with the fact that the sum of two Poisson distributions, the first with parameter \( \lambda \) and the second with
parameter $\mu$ is itself a Poisson distribution, with parameter given by $\lambda + \mu$. From this it is easy to see that if the switching probability is given by $\epsilon(t)$, $\tau \leq t \leq T$ then the cost of power cost delivered between $\tau$ and $T$ is Poisson distributed with parameter $\int_{\tau}^{T} \epsilon(t) dt$. So switching probabilities which vary in time in a deterministic fashion may be effortlessly handled in this framework.

In the next subsection we shall see that even in the event that $\epsilon(t)$ is a random process we can get a nice form for the distribution of power costs, in this case as a mixture of Poisson distributions.

### 2.1.2 Switching Probability a Random Process

We begin with a simple example. Suppose

$$f_k \in \frac{\exp(-\lambda_k)\lambda_k^n}{n!} \equiv Q(\lambda_k) \quad (2.3)$$

where $\lambda_k$ a random process governed by the (very special) geometric random walk

$$\lambda_{k+1} = \begin{cases} 2\lambda_k & \text{with probability } \frac{1}{2} \\ 0 & \text{with probability } \frac{1}{2} \end{cases} \quad (2.4)$$

where $0 \leq \alpha \leq 1$ and where $\lambda_0 = A$. Define a new random variable

$$Z_n = \sum_{k=0}^{n-1} f_k. \quad (2.5)$$

There are two possibilities. Either the sequence of $\lambda_j$ is still nonzero after $n - 1$ steps, which happens with probability $(\frac{1}{2})^{n-1}$, or the first zero in the sequence is at position $k < n - 1$, with probability $1 - (\frac{1}{2})^k$ (we start counting at $k = 0$). In the first case the sum over all $n$ steps of the $\lambda_j$'s is given by

$$A \sum_{j=0}^{n-1} 2^j = A(2^n - 1),$$

in the other cases the sum over all $n$ steps is given by

$$A \sum_{j=0}^{k-1} 2^j = A(2^k - 1).$$
In each of these cases we know how $Z_n$ is distributed, via $Q(2^k - 1), k = 1 \ldots n$ and we conclude that

$$Z_n \in \sum_{k=1}^{n-1} Q(kA)(\frac{1}{2})^k + \left(\frac{1}{2}\right)^{n-1} Q(nA).$$  \hspace{1cm} (2.6)

This is a mixture of Poisson distributions. Note that, in contrast to simple Poisson distributions, the variance of this distribution is greater than its mean.

It turns out that if $\lambda_k$ is driven by a Markov process then the sum $Z_n$ of independent draws of the resulting *Markov-Modulated Poisson sequence* is distributed according to a mixture of Poissons:

$$Z_n \in \int g(\lambda)\Pi_k(\lambda)d\lambda$$  \hspace{1cm} (2.7)

where

$$\Pi_k(\lambda) = \frac{\exp(-\lambda)\lambda^k}{k!}$$  \hspace{1cm} (2.8)

and where $g(\lambda)$ contains the information about how likely a given $\lambda$ parameters is to occur, thus

$$\int g(\lambda)d\lambda = 1.$$

Relatively simple Markov processes can often imply very complicated $g(\lambda)$ weight functions. The model we shall be using in this paper falls into this category.

### 3 A Theoretical Framework for the approximate pricing of weekly swings

There are many different types of swing options. For information about them, see [12], [13], [14]. Here we discuss a simple one. Our simple option has $N$ periods and the swing owner can buy a fixed amount of power at a preset upstrike price at no more than $m$ of these periods. To fix ideas, the $m = 1$ case of this is a “Bermudan” option. We neglect swinging down, latency, and penalty functions. We assume that the upstrike is much greater than the typical “low” price range – this swing is being traded in order that someone can protect their downside. We seek insight.
Let’s begin with some modelling decisions. We suppose that; the low price distribution is unimportant; the high price distribution (on \( dt \) time scale) is fixed (we could relax this and get Compound distributions such as those arising in ruin theory); the time dependent but deterministic component of model is captured by \( \epsilon(t) \); Markov chains drive the load and availability time series, and that these can be combined into a Markov chain driving \( \alpha \) and hence \( \epsilon \). Here the strong nonlinearity of the map between \( \alpha \) and \( \epsilon \) helps us by obscuring most details of the \( \alpha \) process. If we take the same limiting process as in the last section, we find that our price is still Poisson, but now with random \( \lambda \) and weight function \( g(\lambda) \).

### 3.1 Pricing Weekly Swings

We estimate the price of a swing option to buy on-peak power a week at a time. One problem is that on-peak contracts exclude weekends. These missing weekends complicate our Markov modelling, as weather changes and plants get repaired over the weekend.

We can convert this from a problem into an opportunity to simplify. We use the weekends as a grounds for assuming that the corresponding mixture functions \( g_k(\lambda) \) and \( g_{k+1}(\lambda) \) are independent. The weather side of this assumption isn’t too bad – two days is quite a long time in weather modelling. The plant failure part of this assumption is, of course, terrible. The only justification for making it is that, if the option is sold not too far before expiry, any really long-standing plant outage may be already known. Then \( g_k(\lambda) \) can be determined for each week. As in the previous analyses, we ignore interest rates (i.e. set \( r = 0 \)) and risk preferences. This can be relaxed.

We begin with a further simplification. We pretend that the dynamics for a single on-peak day (in North America, 7AM to 11PM) is deterministic given a general daily state. Thus, considering just a single day, we suppose that \( \epsilon(t) \) is variable but deterministic and includes daily load shapes. Thus the poisson parameter for a single day is given by:

\[
\lambda = \frac{1}{16} \int_0^{16} \epsilon(t) dt. \tag{3.1}
\]

We now lumped together all the diurnal variation. Our next assumption is that the dynamics of the daily \( \lambda \) is given by a Markov chain which shuttles between two states, high and low. We denote these by \( H \) and \( L \). The
transition probabilities are then given by $P(H|H) = p$, $P(L|L) = q$ and so $P(L|H) = 1 - p$, $P(H|L) = 1 - q$.

It is now trivial but laborious to show that the resulting mixture of Poisson price distribution is given by the mixture of six different states with the following probabilities:

$$P(\lambda = 5L \equiv \lambda_0) = \frac{1 - p}{2 - p - q} q^4 \equiv p_0, \quad (3.2a)$$

$$P(\lambda = 4L + H \equiv \lambda_1) = \frac{(1 - p)(1 - q)q^2}{2 - p - q} [2q + 3(1 - p)] \equiv p_1, \quad (3.2b)$$

$$P(\lambda = 3L + 2H \equiv \lambda_2) = \frac{(1 - p)(1 - q)q}{2 - p - q} \left[ q(pq + 2(1 - q)(1 - p) + q(1 - q)) + q(p(1 - p) + (1 - p)(1 - q)) + q^2 p \right] \equiv p_2 \quad (3.2c)$$

$$P(\lambda = 2L + 3H \equiv \lambda_3) = \frac{(1 - p)(1 - q)}{2 - p - q} \left[ p(pq + 2(1 - q)(1 - p) + p(1 - p)) + p(q(1 - q) + (1 - p)(1 - q)) + p^2 q \right] \equiv p_3 \quad (3.2d)$$

$$P(\lambda = L + 4H \equiv \lambda_4) = \frac{(1 - p)(1 - q)p^2}{2 - p - q} [2p + 3(1 - q)] \equiv p_4, \quad (3.2e)$$

and

$$P(\lambda = 5L \equiv \lambda_5) = \frac{1 - q}{2 - p - q} p^4 \equiv p_5. \quad (3.2f)$$

Note that the assumption that a week began the day with a high or a low price respectively was obtained from the steady state probabilities of the process

$$P^*(H) = \frac{1 - q}{2 - p - q}, \quad (3.3a)$$

$$P^*(L) = \frac{1 - p}{2 - p - q}. \quad (3.3b)$$

We now price a Bermudan call option struck at $l$ times the price of a single price spike.
Denote the value of a Bermudan call with $N$ exercise points by $B(N)$. Work in “spike” units with the strike then being $l$. Now the value of $B(1)$, a “European” option is:

\[ B(1) = \sum_{k=\text{floor}(l)}^{\infty} (k-l)\pi_k(\lambda) \]

(3.4)

\[ \pi_k(\lambda) = \sum_{j=0}^{5} p_j \frac{\exp(-\lambda_j)\lambda_j^k}{k!}. \]

(3.5)

where the $\lambda_j$ and $p_j$ are given as in (3.2) above. A well known result of Poisson distributions (see Appendix for a detailed derivation) states that

\[ \sum_{k=n+1}^{\infty} (k-l)\pi_k(\lambda) = \int_{0}^{\lambda} \left( \lambda - \frac{l\mu}{n} \right)\pi_{n-1}(\mu)d\mu. \]

(3.6)

This may be used to write

\[ B(1) = \sum_{j=0}^{5} \int_{0}^{\lambda_j} \left( \lambda_j - \frac{l\mu}{m_1} \right)\pi_{m_1-1}(\mu)d\mu; \]

(3.7)

\[ m_1 = \text{floor}(l). \]

(3.8)

Now let’s begin to calculate the rules for early exercise. At the second-last exercise date the option holder may hold the security or exercise it. The early exercise boundary is the point at which the option holder is indifferent between these alternatives. Denote the indifference point at 2 points to go by $\Theta(2)$. Then clearly $\Theta(2) - l = B(1)$. If the price is below $\Theta$ the rule says to hold; if above, exercise. Written another way,

\[ \Theta(2) = B(1) + l; \]

(3.9)

\[ m_2 = \text{floor}(\Theta(2)). \]

(3.10)

Proceeding in a similar fashion, we may obtain the following recursion relations:

\[ m_N = \text{floor}(\Theta(N)) \]

(3.11)

\[ \Theta(N) = B(N-1) + l \]

(3.12)
\[ B(N) = \sum_{k=0}^{m_N} B(N-1)\pi_k(\lambda) + \sum_{k=m_N+1}^{\infty} (k-l)\pi_k(\lambda) \]  \hspace{1cm} (3.13)

Note that it is easy to add interest rates to the above model. It is easy to generalize the calculation to prices drawn from mixtures of Poisson variables. It is easy, but tedious, to add time varying \( g(\lambda) \). It all works more or less the same way. The only hard part is determining the \( g(\lambda) \).

Now let's consider some more interesting options. Consider a swing option \( S(N, k) \) where we can swing up \( k \) of the next \( N \) dates. This problem is also solved using backwards recursion beginning with \( S(1) \). The early exercise boundary is harder to work out but is given by:

\[ \Theta(N, k) - l + S(N - 1, k - 1) = S(N - 1, k), \]  \hspace{1cm} (3.14)

\[ m_{Nk} = \text{floor}(\Theta(N, k)), \]  \hspace{1cm} (3.15)

\[ S(N, k) = \sum_{k=0}^{m_{Nk}} S(N - 1, k)\pi_k(\mu) + \sum_{k=m_{Nk}+1}^{\infty} \left( k - l + S(N - 1, k - 1) \right)\pi_k(\mu). \]  \hspace{1cm} (3.16)

4 Outlook

We are excited by the new avenue for realistic and efficient pricing models opened up by the above insights. For the reader who is unhappy with all the assumptions that need to be made for this process to work, we would again emphasize that our goal in doing this is the quick approximate computation of early exercise boundaries. With these in hand, Monte Carlo simulation of more realistic models can be envisaged. The resulting prices will be an underestimate of the true options price, as the exercise boundaries computed using these techniques will not be optimal for the unsimplified models. However, a good starting point for optimization of the early exercise region of such models will have been obtained. We also emphasize that a great deal of work remains to be done. Most crucially, we need to determine how to assign to each set of parameters in our complicated \( \alpha \) model of \([4]\) a \( g(\lambda) \) describing our simplified mixture of Poissons model.
References


A result about Poisson partial sums

Let
\[ \pi_k(\mu) = \frac{\mu^k \exp(-\mu)}{k!} \]

Then
\[ \sum_{k=0}^{n} \pi_k(\mu) = 1 - \int_{0}^{\mu} \pi_n(\lambda) d\lambda. \]

with corollary
\[ \sum_{k=n+1}^{\infty} \pi_k(\mu) = \int_{0}^{\mu} \pi_n(\lambda) d\lambda. \]

Proof:
\[ \frac{d\pi_k(\mu)}{d\mu} = \frac{k \mu^{k-1} \exp(-\mu)}{k!} - \frac{\mu^k \exp(-\mu)}{k!} = \pi_{k-1}(\mu) - \pi_k(\mu) \]

if \( k \geq 1 \), so
\[ \frac{d}{d\mu} \sum_{k=0}^{n} \pi_k(\mu) = \frac{d}{d\mu} (\exp(-\mu)) + \frac{d}{d\mu} \sum_{k=1}^{n} \pi_k(\mu) \]
\[ = -\pi_0(\mu) + \sum_{k=1}^{n} \left[ \pi_{k-1}(\mu) - \pi_k(\mu) \right] = -\pi_n(\mu) \]

since the series telescopes. Also
\[ \sum_{k=0}^{n} \pi_k(0) = 1. \]

Thus if we suppose that
\[ y(\mu) = \sum_{k=0}^{n} \pi_k(\mu), \]

then we have the initial value problem
\[ \frac{dy}{d\mu} = -\pi_n(\mu); \quad y(0) = 1, \]
which has solution

\[ y(\mu) = \sum_{k=0}^{n} \pi_k(\mu) = 1 - \int_{0}^{\mu} \pi_k(\lambda) d\lambda \]

The corollary follows immediately from the normalization condition

\[ \sum_{k=0}^{\infty} \pi_k(\mu) = 1. \]