The Early Exercise Region for Bermudan Options on Multiple Underlyings

Jeff Kay, Matt Davison, and Henning Rasmussen
jkay@uwo.ca, mdavison@uwo.ca, hrasmuss@uwo.ca

Abstract

In this paper we investigate the early exercise region for Bermudan/American options on multiple underlying assets. We present a set of analytical validation results for the multi-asset early exercise region which can be used as a means of validating pricing techniques without having to depend on “standard” results presented elsewhere. We find an intersection point in the multi-asset early exercise region – when all strike prices are identical – whose implication is such that for any asset price pair below this point early exercise is always optimal, and develop an approximation to this point. When the strike prices are not all equal, we show that three separate cases exist for the early exercise region. For a Bermudan put on two assets we present these cases and show that there exists a critical point \( \phi \) in which the boundaries of the two asset early exercise region bifurcate.

\footnote{Department of Applied Mathematics, The University of Western Ontario, London, Canada N6A 5B7}
1 Introduction

The pricing of American options on a single underlying asset following geometric Brownian motion has been extensively studied. As well, the early exercise regions for these options on dividend paying calls and puts has been thoroughly investigated. In recent years, attention has turned to the pricing of American options on multiple underlying assets. In 1987 Johnson[9] derived the European price on multiple underlying assets in terms of multivariate cumulative normals. However, in the multi-variate American/Bermudan case, analytic solutions remain elusive in much same way as they do in the standard American case. Since Johnson, a number of practitioners have developed numerical techniques for pricing multi-asset options, including Barraquand and Martineau [3], Broadie and Glasserman [4] and Longstaff and Schwartz[12], among others. Fu et al.[5] and Garcia[6] both provide excellent reviews of the relevant material.

To date however, few papers have investigated the early exercise regions of American options on multiple underlying assets, sometimes termed rainbow options. Tan and Vetzal[14] in 1995 examined the early exercise region of American options on the maximum (put) and minimum (call) of two underlying assets. As well Ibanez and Zapatero[8] in their pricing paper provided results for the early exercise region of a American call option on two assets.

Many numerical methods for pricing American options proceed by considering only a finite, though possibly large, set of early exercise opportunities. As such they are therefore actually pricing American options by approximating them as Bermudan options. Much analytic work has been presented on the structure, in varying limits, of the one dimensional American solution, by using the structure of the moving boundary partial differential equation[2][11][13][7].

Herein we investigate the early exercise region of Bermudan options on multiple underlying assets following geometric Brownian motion. Our contribution is novel in three significant ways. First, we present results for two cases; for a symmetric case in which all strike prices are equal as well as for an asymmetric case in which not all strike prices agree. Examination of numerically determined early exercise regions shows an intersection (point) between the early exercise boundaries of the symmetric case. We also see the emergence of three distinct early exercise regions in the asymmetric case. Which of these three distinct regions occurs depends on a critical point which we determine. Secondly, we develop a set of techniques and analytic formulæ for validation of the early exercise regions based on Bermudan option structure. Among them, we find an approximation to the intersection point of the early exercise boundaries in the symmetric d asset max-max put and we present an analytic formula for the critical point in the asymmetric case of
a two asset max-max put. Furthermore, we show an example of where the
Bermudan and American exercise regions differ but agree in the appropriate
limit. Lastly, we present the early exercise regions for Bermudan options on
three assets and show the effect of multiple exercise dates.

The structure of this paper is as follows:

In section 2 we define and describe multi-asset Bermudan options and present
numerical pricing techniques for solving them. We utilize these methods to
illustrate the utility of our validation results and obtain graphical representa-
tions of the early exercise boundaries.

In section 3 we define symmetric Bermudan options and present the early ex-
ercise regions for two asset Bermudan options and corresponding validation
results.

Section 4 presents asymmetric Bermudan options and describes the three
possible early exercise regions which arise under asymmetric condition for
varying option parameter values. We show the existence of a critical point
in this case and find an analytic formula for it in the two asset case.

In section 5 we generalize many of these ideas to arbitrarily high dimensions.
We follow the put results with some two and three asset call results in section
6 and we concluded finally in section 7 with a discussion of the results we
present.

2 Multi-Asset Options

A number of methods can used to price multi-asset options. Herein we em-
ploy two techniques to obtain the early exercise boundaries of Bermudan
options on multiple underlying assets. Both techniques utilize a dynamic
programming methodology, working backwards from expiry simultaneously
computing the option value and the corresponding early exercise bound-
aries at each exercise opportunity. Here we present the first of the pricing
techniques. A lattice Monte Carlo method is described in appendix B to
facilitate presenting the bulk of our results sooner.

2.1 Iterated Integral Method

Consider an option with value $V$ with $N$ exercise opportunities $t_1, t_2, \ldots, t_N$
on $d$ geometric Brownian motion assets $S_1, S_2, \ldots, S_d$, where $t_0$ is contract
initiation, $t_N$ expiry and $S_d$ is the $d^{th}$ asset.

At expiry, the value of the option $V$ is given by the payoff function

$$V_N(S_1, \ldots, S_d) = H(S_1, \ldots, S_d).$$
At the last early exercise opportunity $t_{N-1}$, the value of the option is the same as a European option on $d$ assets and is given by

$$ V_{N-1}(S_1,\ldots,S_d) = \int_0^\infty \cdots \int_0^\infty H(S'_1,\ldots,S'_d) \mathcal{G}(S_1,\ldots,S_d;S'_1,\ldots,S'_d)\,dS'_1 \cdots dS'_d $$

(1)

where $\mathcal{G}(S_1,\ldots,S_d;S'_1,\ldots,S'_d)$ is the multi-dimensional Black-Scholes European Green’s Function.

For any other exercise opportunity $t_i, i \in N-2,\ldots,0$, we can view the value of the option between the $i^{th}$ and $(i + 1)^{st}$ exercise opportunity as a European option on the $i^{th}$ exercise opportunity with payoff

$$ H(S_1,\ldots,S_d,V_{i+1}(S_1,\ldots,S_d)). $$

The corresponding option value is given by

$$ V_i(S_1,\ldots,S_d) = \int_0^\infty \cdots \int_0^\infty H(S'_1,\ldots,S'_d,V_{i+1}(S_1,\ldots,S_d)) \mathcal{G}(S_1,\ldots,S_d;S'_1,\ldots,S'_d)\,dS'_1 \cdots dS'_d. $$

(2)

At each exercise opportunity $t_i$ there exists corresponding exercise boundaries $\phi^j_i$ for each asset $S_j$ which divide the early exercise region at each opportunity into subregions where it is either optimal to exercise the option with respect to one of the underlying assets $S_j$ or to hold the option – a continuation subregion – until at least the next early exercise opportunity.

2.1.1 Two Asset Case

In this section we restrict ourselves to an option on two assets whose value at expiry is given by the max-max payoff,

$$ V_N(S_1,S_2) = H(S_1,S_2,0) = \max(\max(K_2-S_2,K_1-S_1),0). $$

The value of the option at the last early exercise opportunity is simply the European option value given by

$$ V_{N-1}(S_1,S_2) = \int_0^\infty \int_0^\infty H(S_1,S_2)\mathcal{G}(S_1,S_2;S'_1,S'_2)\,dS'_1\,dS'_2. $$
where the Green’s function [15] for a two asset European option following GBM is given by

\[
G(S_1, S_2; S_1', S_2') = \frac{e^{-r(T-t)}}{2\pi(T-t)(1-\rho^2)^\frac{1}{2}\sigma_1\sigma_2} e^{\frac{-|\alpha_1^2-2\rho\alpha_1\alpha_2+\alpha_2^2|}{2(1-\rho^2)}},
\]

with

\[
\alpha_i = \frac{1}{\sigma_i\sqrt{T-t}} \left( \ln \frac{S_i}{S_i'} + (r - \frac{\sigma_i^2}{2})(T-t) \right).
\]

For all other exercise opportunities the option value at the \(i^{th}\) opportunity is given by

\[
V_i(S_1, S_2) = \int_0^\infty \int_0^\infty H(S_1, S_2, V_{i+1}(S_1, S_2)) G(S_1, S_2; S_1', S_2') dS_1' dS_2', \quad (3)
\]

where the corresponding early exercise boundaries are given by

\[
K_1 - \phi_{N-1}^1(S_2) = V_{N-1}(\phi(S_2), S_2), \quad K_2 - \phi_{N-1}^2(S_1) = V_{N-1}(S_1, \phi(S_1)). \quad (4)
\]

The option value is obtained by starting at expiry and recursing backward, solving for the option value at each early exercise opportunity and the corresponding early exercise boundaries.

The option value given by equation 3 can be reduced analytically to a single dimensional integral. The remaining expression cannot be further evaluated analytically and thus we rely upon numerical integration techniques. We use Gauss-Legendre integration [1].

Gauss-Legendre is an obvious choice for the low order problems given that we have an integral which is readily transformed into the appropriate form with correct weight functions. Furthermore Gauss-Legendre integration avoids evaluation of the integrand at the endpoints where we have singularities.

3 Symmetric Bermudan Put Options on Multiple Assets

We now restrict our focus to what we call the symmetric case. We consider Bermudan put options where the corresponding strike prices of all underlying
assets are the same, \( K_1 = K_2 = \ldots = K_d \). In addition, all corresponding volatilities are the same. Without the volatility restriction such options are sometimes called outperformance options or options on the minimum of \( d \) assets. To maintain consistent terminology with the asymmetric case, we instead consider them in terms of the maximum payoff relative to the strike. The value of this option at expiry is given by the payoff

\[
H(S_1, \ldots, S_d) = \max (\max (K - S_1, 0), \ldots, \max (K - S_d, 0)) = \max (K - \min (S_1, \ldots, S_d))
\]

For clarity and ease of explanation, we further restrict our focus to a Bermudan put option on two underlying assets with a single early exercise date and expiry. Our approach and pricing techniques can be employed for higher dimensional problems as well. In section 5 we generalize many of these results to arbitrary \( N \) dimensions.

Employing the iterated integral method using 1000 Gauss-Legendre integration points, we solve for the early exercise boundaries of the two asset max-max problem with parameters given in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_1 ) (yrs(^{-1/2} ))</td>
<td>0.4</td>
</tr>
<tr>
<td>( \sigma_2 ) (yrs(^{-1/2} ))</td>
<td>0.4</td>
</tr>
<tr>
<td>( r ) (yrs(^{-1} ))</td>
<td>0.1</td>
</tr>
<tr>
<td>( T ) (yrs)</td>
<td>0.5</td>
</tr>
<tr>
<td>( K ) ($)</td>
<td>10</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 1 shows the early exercise boundaries obtained for the final early exercise surface with these parameters. The early exercise region is divided by the boundaries into three distinct regions; exercise with respect to \( S_1 \), exercise with respect to \( S_2 \), and a hold or continuation region. There is an intersection point of the early exercise boundaries, such that for any stock pair below this intersection point \((S, S)\) it is optimal to exercise the option.

As noted by Tan and Vetzal[14], there exists a “wedge” region \( P_1(S_i, S_j) \) where the intrinsic value of the option may be higher than in other areas \( P_2(S_i, S_j) \) where it is optimal to exercise the option with a lower intrinsic value and yet one does not exercise the option. This seeming contradiction can be explained in the wedge region as follows. Consider a point deep in the wedge region \( P_1(S_i, S_j) \). Neglecting drift temporarily; as the option holder we have a three in four chance that one of the stocks falls placing the option
holder within one of the neighbouring exercise regions, increasing the intrinsic value and only a one in four chance that both stocks increase in value thus moving us further out of the wedge region into the hold (continuation) region decreasing the intrinsic value. While the “three in four” argument also holds below the intersection point there is also a balance between interest on immediate exercise and future beneficial stock price evolution. The intersection is a balance point between these counteracting forces.

3.1 The Effect of Parameter Changes

The effect of changes to the various parameters on the early exercise boundaries was investigated. Using the same parameters as those in Figure 1, Figure 2 shows the parameter change effects when \( r = 0.2 \) and when \( \sigma_1 = \sigma_2 = 0.6 \).

The changes to the exercise boundary are summarized in Table 2

3.2 Validation Formulae

In the symmetric case the intersection point always falls along the line \( S_1 = S_2 \). This is because under the symmetry conditions the option value is symmetric about \( S_1 = S_2 \) as are the two early exercise boundaries and thus
they can only intersect along this line. Using this knowledge we sought an analytic approximation to the intersection point $(S, S)$.

We determine the early exercise boundaries by balancing the present value of immediate exercise to that of holding the option. We can also think of this as balancing the future benefit due to interest of exercising the option with the future benefit that we may obtain from the option (primarily) due to random evolution of the stock price. Since this is true for the early exercise boundaries, it is also true for the intersection point. For a Bermudan put option on two underlying assets with exercise opportunity spacing $\Delta T$ we balance the benefit from interest in any one period $\Delta T$ against the expected benefit of holding the option due to volatility. Thus starting with

$$(K - S)(e^{r\Delta T} - 1) = E[\max (\max (K_2 - S_2, K_1 - S_1), 0)] - (K - S),$$

they can only intersect along this line. Using this knowledge we sought an analytic approximation to the intersection point $(S, S)$.
Table 3: Comparison of the Symmetric Two Asset Case Results Against the Validation Formulae

<table>
<thead>
<tr>
<th>$\sigma$, $r$, T, K</th>
<th>$S = \frac{Kr\sqrt{\pi\Delta T}}{\sigma}$</th>
<th>Intersection Point</th>
<th>Tail Value</th>
<th>True Tail Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4, 0.1, 0.5, 10</td>
<td>2.2156</td>
<td>2.1930</td>
<td>8.4334</td>
<td>8.4336</td>
</tr>
<tr>
<td>0.6, 0.1, 0.5, 10</td>
<td>1.4771</td>
<td>1.4614</td>
<td>7.3981</td>
<td>7.4093</td>
</tr>
<tr>
<td>0.2, 0.1, 0.5, 10</td>
<td>4.4311</td>
<td>4.3584</td>
<td>9.4415</td>
<td>9.4415</td>
</tr>
<tr>
<td>0.4, 0.2, 0.5, 10</td>
<td>4.4311</td>
<td>4.3330</td>
<td>8.9495</td>
<td>8.9495</td>
</tr>
<tr>
<td>0.4, 0.05, 0.5, 10</td>
<td>1.1077</td>
<td>1.1042</td>
<td>7.9453</td>
<td>7.9455</td>
</tr>
<tr>
<td>0.4, 0.1, 0.5, 12</td>
<td>2.6586</td>
<td>2.6331</td>
<td>10.1202</td>
<td>10.1203</td>
</tr>
<tr>
<td>0.4, 0.1, 0.5, 8</td>
<td>1.7725</td>
<td>1.7490</td>
<td>6.7468</td>
<td>6.7469</td>
</tr>
</tbody>
</table>

and after some manipulations – the full details of which are contained in the appendices – we obtain to leading order

$$S \approx \frac{Kr\sqrt{\pi\Delta T}}{\sigma}.$$  \hspace{1cm} (5)

In section 5 we present a similar result valid for the d-dimensional symmetric case.

In all our tests the intersection point approximation accurately predicted the intersection point of the two asset Bermudan put option. Table 3 shows this comparison for various values of the parameters. Furthermore, the location of the intersection point determined using Equation 5 is consistent with the observed changes to the early exercise boundaries shown in Table 2.

Conventional wisdom among traders states for an American option on two assets with the same strike, never exercise the option when the two assets have the same price. Although the intersection point shown here seem to disagree with this maxim, if $\Delta T \to 0$ as in the American case then from Equation 5 $S \to 0$ moving the intersection point to the origin, confirming the traders intuition.

As we mentioned in our introduction, the intersection point and early exercise region along the line $S_1 = S_2$ are one case where the American and Bermudan results differ. In fact although they never mentioned the intersection point explicitly, the early exercise results (Exhibit 1) of Tan and Vetzel using a Crank-Nicholson finite difference scheme with 52 early exercise dates to approximate an American option depicts an intersection point. This one example where using a Bellman approach can potentially lead to incorrect results. The situation is made worse if the correlation between stocks is near one.
3.2.1 Tail Validation

Also sought was a method for validating the tail ends of the early exercise boundaries, when either \( S_1 \) or \( S_2 \) tend to infinity. From intuition we know that if one of the stocks (\( S_1 \)) tends to infinity, then it is so far out of the money that near expiry it is highly unlikely it will be subject to a beneficial price movement that will place it back in the money. In this situation we can neglect this stock, leaving us with an option on the remaining stock (\( S_2 \)). In the two asset case when one asset tends to infinity the two asset Bermudan option value tends to a standard Bermudan Option on a single asset

\[
V(S_1, S_j) \to V(S_j) \quad \text{as} \quad S_1 \to \infty,
\]

where the tail value is given by the nonlinear equation

\[
K_j - S_j = V(S_j).
\]  

(6)

The tail values obtained graphically from the two asset Bermudan early exercise boundaries are shown in Table 3 along with the tail values obtained from equation 6 for the same parameters.

3.3 The Effect of Additional Early Exercise Dates

Thus far we have explicitly considered a Bermudan put option on multiple underlying assets with only one early exercise date. We can extend many of the validation results to the case of multiple early exercise dates. The pricing techniques we employed require no extension as they are already generalized to both multiple dimensions and exercise dates.

The simplest validation technique to extend is the tail boundary validation. We have shown that the tail of the two asset early exercise boundaries can be verified by comparing them to early exercise point of an equivalent European option. For multiple exercise dates the tail values for the two asset early exercise boundaries can be compared to the early exercise points at the corresponding early exercise opportunities of an equivalent one dimensional Bermudan option (or two equivalent single asset Bermudan options in the case of an asymmetric two asset Bermudan option).

In the case of multiple exercise dates, it is the tail value at the last early exercise opportunity that is given essentially analytically, requiring only the solution of a nonlinear equation. This equation can be solved to arbitrarily high accuracy and thus we view it as exact. All other tail values must be compared to early exercise points from an equivalent single asset Bermudan
option, the results of which are only as good as the method employed to obtain the Bermudan option value. That said there are many single asset methods available which are suitable for testing and comparison purposes.

Our approach to obtain the intersection point approximation formula is not easily extended to multiple exercise dates. We can however build some intuition about how the intersection point behaves as we recurse back though additional early exercise dates.

We know that the value of a Bermudan option lies between that of a European option and an American option with equivalent parameter values. Thus we know that an option over some time period $T$ with $i$ early exercise opportunities is worth more than or equivalent to one with $j$ opportunities, $i > j$. Thus

$$V_i \geq V_j, \forall i > j.$$  \hspace{1cm} (7)

The intersection point of a two asset option is given by the nonlinear equation

$$K - \phi_j = V(\phi_j),$$  \hspace{1cm} (8)

where $j$ is the last early exercise opportunity. Therefore it easily follows that

$$K - \phi_i \geq V_j = K - \phi_j,$$

$$\phi_i \leq \phi_j.$$  \hspace{1cm} (9)

Figure 3 shows the early exercise region for a Bermudan put option with ten exercise dates (nine early). Figure 4 shows a two dimensional side view of the early exercise surfaces. From a number of our tests, it appears that the intersection point moves a negligible amount as we move backward through a given set of early exercise opportunities and thus our intersection point can perhaps be employed to validate the intersection point at any early exercise opportunity.

4 Asymmetric

The asymmetric case is defined to occur when not all of the strike prices are equal. We again consider the two asset Bermudan max-max put options where $K_1 \neq K_2$ with payoff
Exercise Dates

Figure 3: Effects Of Multiple Early Exercise Dates on the Early Exercise Region

\[ H(S_1, S_2) = \max \left( \max (K_1 - S_1, K_2 - S_2), 0 \right). \]

We assume \( K_2 > K_1 \) and define \( K \equiv K_2 - K_1 \). We solved for the early exercise boundaries of the Bermudan put option with one exercise (expiry) opportunity remaining using the iterated integral method with 1000 Gauss-Legendre integration points and option parameters given in Table 4.

**Table 4: Asymmetric Two asset Bermudan Put Option Parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_1 ) (( yrs^{-\frac{1}{2}} ))</td>
<td>0.4</td>
</tr>
<tr>
<td>( \sigma_2 ) (( yrs^{-\frac{1}{2}} ))</td>
<td>0.4</td>
</tr>
<tr>
<td>( r ) (( yrs^{-1} ))</td>
<td>0.1</td>
</tr>
<tr>
<td>( T ) (( yrs ))</td>
<td>0.5</td>
</tr>
<tr>
<td>( K_2 ) ($)</td>
<td>10</td>
</tr>
<tr>
<td>( K_1 ) ($)</td>
<td>5</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0</td>
</tr>
</tbody>
</table>

The resulting early exercise region is shown in Figure 5. As one can see the two exercise regions are now completely separated or bifurcated and the hold region extends onto the \( S_2 \) axis.
In the single asset case if the stock ever reaches zero one is guaranteed to exercise the option since the payoff can never improve. Indeed this intuition is all that is necessary to show that early exercise exists in the American put option. Now in the two asset case we can have the situation where one of the stocks has fallen to zero and yet there is a region where one would hold the option.

There is a simple explanation that provides the needed insight under these conditions. In the above example $K_1 = 5$ and $K_2 = 10$. Thus when $S_1 = 0$ we have a guaranteed payoff of $K_1 = 5$. Now since $S_1$ can no longer fluctuate we need only look at what happens to $S_2$. If $S_2 = 5$ the immediate payoff with respect to $S_2$ is the same as the guaranteed payoff with respect to $S_1$. Thus a rational investor would be indifferent between the two. Stock 2 has the potential to fall lower thus increasing our possible payoff with respect to $S_2$. If $S_2$ increases in value and the corresponding payoff decreases, we still have the guaranteed payoff of $K_1$. Thus neglecting interest, at worst we lose nothing by holding but have the potential to gain a greater payoff with respect to $S_2$. In fact by holding one forgoes the interest on immediate exercise in favour of the potential for volatility movements in $S_2$ and one would only ever hold if the size of the volatility movements in any one time period $\Delta T$ were greater than the interest gained, which is generally the case with typical option parameters.

Three cases exist for the asymmetric strikes: the fully bifurcated case of Figure 5, an unbifurcated case (Figure 6a) where the exercise boundaries again intersect though no longer along the line of symmetry $S_1 = S_2$ and
Due to the loss of symmetry the asymmetric results cannot be validated using the intersection point formula of the symmetric case. Under certain circumstances the asymmetric cases exhibit separation of the exercise boundaries along one of the geometric boundaries (axes). This behaviour occurs when one strike price is significantly larger than the other strike price(s). For example if $K_2 > K_1$ when $|K_2 - K_1| > \phi$, a critical point, separation along the $S_2$ boundary occurs as was shown in Figure 5. We wish to validate the $S_2$ intercepts of the early exercise boundaries for the asymmetric case when the boundaries are fully bifurcated.

To validate the asymmetric results in this case we investigate what happens when $S_1 \to 0$. If $S_1 = 0$ then we are left with an option in only one stock, $S_2$, which has a slightly novel payoff

$$H(S) = \max(K_2 - S_2, K_1).$$

As above, the reason for this is that once a stock reaches zero it remains and can never return to any positive asset value. At this point said stock is now deterministic and no source of uncertainty remains with respect to it.

Figure 5: Simple Case Asymmetric Early Exercise Region

the emergence of bifurcated exercise regions shown in Figure 6b.

4.1 Asymmetric Validation Formulae

Due to the loss of symmetry the asymmetric results cannot be validated using the intersection point formula of the symmetric case. Under certain circumstances the asymmetric cases exhibit separation of the exercise boundaries along one of the geometric boundaries (axes). This behaviour occurs when one strike price is significantly larger than the other strike price(s). For example if $K_2 > K_1$ when $|K_2 - K_1| > \phi$, a critical point, separation along the $S_2$ boundary occurs as was shown in Figure 5. We wish to validate the $S_2$ intercepts of the early exercise boundaries for the asymmetric case when the boundaries are fully bifurcated.

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$$H(S) = \max(K_2 - S_2, K_1).$$

As above, the reason for this is that once a stock reaches zero it remains and can never return to any positive asset value. At this point said stock is now deterministic and no source of uncertainty remains with respect to it.
Figure 6: Early Exercise Boundaries in the Two Asset Asymmetric Bermudan Put

The option is thus left with only one stock \( (S_2) \) as a source of uncertainty. However an option with only one uncertain stock is truly a one dimensional option, though in this case one with a non-standard payoff. Thus we postulate that when \( S_1 \) tends to zero, the two asset option tends to this novel one asset option.

Thus we have an option with a minimum guaranteed payoff of \( K_1 \) with the possibility of obtaining \( K_2 - S_2 \). The value of this option is given by

\[
V_{1D}(S,t) = \int_0^\infty \max(K_2 - S', K_1) G(S;S')dS',
\]

where \( G(S;S') \) is the Black-Scholes Green’s function

\[
G(S;S',T-t) = \exp\left[-\frac{r(T-t)}{2\sigma^2} \frac{1}{S'} \exp \left[ -\frac{\left(\ln\left(\frac{S}{S'}\right) + (r - \sigma^2(T-t))^2}{2\sigma^2(T-t)}\right)\right]\right].
\]

With this novel one dimensional option the discontinuity in the final exercise surface payoff – also known as the “hockey stick” – no longer occurs at \( K_2 \) but instead occurs at \( K \). The point at which one switches between preferring the payoff \( K_2 - S \) to the guaranteed payoff \( K_1 \) occurs at \( K \). Thus we can simplify the above integral by integrating each payoff over the appropriate region as follows.

\[
V_{1D}(S,t) = \int_0^K (K_2 - S')G(S;S')dS' + \int_K^{\infty} K_1 G(S;S')dS',
\]

where

\[
K = K_2 - K_1 \text{ and } K_2 > K_1.
\]
Making appropriate substitutions and integrating we obtain

\[ V_{1D}(S, t) = e^{-r(T-t)}[(K_2 - K_1)N(-d_2) + K_1 - Se^{r(T-t)}N(-d_1)], \quad (10) \]

where

\[
\begin{align*}
    d_1 &= \frac{\ln \frac{S}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}} \\
    d_2 &= d_1 - \sigma \sqrt{T-t}. \\
\end{align*}
\quad (11)
\]

This is the option value for a one dimensional European option with payoff \( \max(K_2 - S, K_1) \).

It is well known\(^8\) that in the standard European put case only a single root exists for

\[ K - S = V(S). \]

For our novel one dimensional option the observed behaviour of the two asset asymmetric case requires; two roots, a single tangency root and no roots, corresponding to fully bifurcated, bifurcating and intersecting early exercise boundaries respectively. Figure 7 shows these three cases for the two asset asymmetric Bermudan option with one remaining exercise date. Also shown is the novel single asset option. The option value is plotted against the payoff (hockey stick) and shows the existence of the requisite number of roots for each case.

For the unbifurcated case although the plot of the early exercise boundaries appear to show the intersection of the boundaries with the \( S_2 \) axis we require that no roots exist. For all values below the intersection point it is optimal to exercise the option. The continuation of the boundaries graphically beyond the intersection point is a result of the nonlinear equation that we solve to obtain the boundaries but has no financial implication below the intersection point. That is, below this point boundaries are not financially meaningful; the intercepts that we see are merely ghost images, a by-product of the equation we solve. Hence no roots should exist in the novel one asset option which indeed is the case.

We solve for these points in the same way that we solve for the early exercise boundaries elsewhere by solving a set of nonlinear algebraic equations which balance the expected value of continuation to that of the immediate value of exercise.
Figure 7: Early Exercise Boundaries of the Asymmetric Option Contrasted with the Roots of the Novel One Asset Option
In the two dimensional framework one would normally solve the following equations:

\[ K_2 - \phi(S_1) = V(S_1, \phi(S_1)), \]
\[ K_1 - \theta(S_2) = V(\theta(S_2), S_2). \]

However since \( S_1 = 0 \) at the point of interest the above equations simplify to,

\[ K_2 - \phi(0) = V(0, \phi(0)), \]
\[ K_1 = V(0, S_2). \]

The analytic behaviour of \( V(0, S) \) in the \( S_1 \to 0 \) limit is not obvious. Previously we argued that when \( S_1 = 0 \) we have the novel one asset option \( V_{1D}(S) \). Thus we postulate that

\[ V(S_1, S_2) \to V_{1D}(S) \quad \text{as} \quad S_1 \to 0. \]

Using this novel option we can obtain the intercept points \((\phi, \theta)\) of the early exercise boundaries when \( S_1 = 0 \) by solving

\[ K_2 - \phi = V_{1D}(\phi), \]
\[ K_1 = V_{1D}(\theta). \] (12)

Table 6 shows a comparison of the actual intercept values \((\phi, \theta)\) obtained via graphical inspection and the “exact” values which were computed by solving the nonlinear equations (12) which utilize the novel one asset option. We tested these for various parameters a sample of which are shown in Table 5.

**Table 5: Asymmetric Intercept Point Parameter Values**

<table>
<thead>
<tr>
<th>Number</th>
<th>Parameter Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \sigma = .4, r=0.1, T=0.5, K_1=5, K_2=10 )</td>
</tr>
<tr>
<td>2</td>
<td>( \sigma = .6, r=0.1, T=0.5, K_1=5, K_2=10 )</td>
</tr>
<tr>
<td>3</td>
<td>( \sigma = .2, r=0.1, T=0.5, K_1=5, K_2=10 )</td>
</tr>
<tr>
<td>4</td>
<td>( \sigma = .4, r=0.2, T=0.5, K_1=5, K_2=10 )</td>
</tr>
<tr>
<td>5</td>
<td>( \sigma = .4, r=0.1, T=0.5, K_1=2, K_2=10 )</td>
</tr>
</tbody>
</table>

As can be seen the nonlinear equations (12) accurately predict the location of the intercepts of the early exercise boundaries in the bifurcated asymmetric option case. This is further evidence supporting our conjecture about what occurs as \( S_1 \to 0 \).
Table 6: Comparison of Asymmetric Intercept Points to Validation Formulæ

<table>
<thead>
<tr>
<th>Number</th>
<th>$\theta$ Exact</th>
<th>$\phi$ Exact</th>
<th>$\theta$ Exact</th>
<th>$\phi$ Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.6479</td>
<td>4.6692</td>
<td>5.6409</td>
<td>4.6821</td>
</tr>
<tr>
<td>2</td>
<td>6.7561</td>
<td>4.1589</td>
<td>6.7306</td>
<td>4.1421</td>
</tr>
<tr>
<td>3</td>
<td>5.0496</td>
<td>4.9707</td>
<td>5.0511</td>
<td>4.9741</td>
</tr>
<tr>
<td>4</td>
<td>5.1006</td>
<td>4.9429</td>
<td>5.1020</td>
<td>4.9486</td>
</tr>
<tr>
<td>5</td>
<td>10.6133</td>
<td>6.9189</td>
<td>10.5976</td>
<td>6.9203</td>
</tr>
</tbody>
</table>

4.2 Asymmetric Bifurcation Point

Initially through experimentation, the only two regimes that were encountered were the bifurcated case or the unbifurcated case. The question arose “was there a case in which the two exercise boundaries would bifurcate along one of the axis of the early exercise region?” If this phenomenon does indeed occur, and if one could find an analytic approximation to this point, then it would be an additional method of validation.

As mentioned earlier, for a certain set of parameter values an early exercise surface can be obtained which shows the evolution or beginning of the bifurcated early exercise region. Using equation 10 for the novel one dimensional option and the nonlinear equations 12 we obtained an equation which pinpoints for a given set of parameters the point at which bifurcation occurs.

Clearly if the two early exercise boundaries do coincide at some bifurcation point $(0, \hat{\phi})$, then at that point the early exercise points obtained from equation 12 $(\phi, \theta)$ must be the same. Thus we have

$$\hat{\phi} = \phi = \theta.$$  

Substituting this into equation 12 we now have

$$K_2 - \hat{\phi} = V_{1D}(\hat{\phi}),$$
$$K_1 = V_{1D}(\hat{\phi}).$$

Since the right hand sides of both equations are the same we now obtain

$$\hat{\phi} = K_2 - K_1.$$  \hspace{1cm} (13)

Therefore the early exercise boundaries bifurcate along the $S_2$ axis at the point $(0, K_2 - K_1)$. While we now know the coordinates of the point of bifurcation in terms of the parameters $K_1, K_2$ the early exercise boundaries do not coincide and bifurcate from the $S_2$ axis for all values of $K_1, K_2$. Indeed, only for certain pairs of $K_1, K_2$ values does the early exercise region bifurcate, but always at the point $(0, K_2 - K_1)$. 

19
Given $K_2, \sigma_2, \sigma_1, r, T - t$ we can find an equation of the form

$$K_1 = \alpha K_2,$$

where $\alpha$ is a function of $\sigma$, $r$, and $T - t$ that determines for which pairs of $(K_2, K_1)$ bifurcation occurs.

Since we know that $\hat{\phi} = K_2 - K_1$ and that

$$K_1 = V_{1D}(\hat{\phi})$$

at the bifurcation point, substituting equation 13 into equation 14 yields

$$K_1 = Ke^{-r(T-t)N\left(\frac{-(r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}\right)} + K_1 - KN\left(\frac{-(r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}\right),$$

where

$$K = K_2 - K_1,$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$  

Solving for $K_1$ we obtain

$$K_1 = K_2 \left[ \frac{e^{-r(T-t)}N(b_2) - N(b_1)}{e^{-r(T-t)}(N(b_2) - 1) + 1 - N(b_1)} \right], \quad (15)$$

where

$$b_1 = \frac{-(r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

$$b_2 = b_1 + \sigma\sqrt{T - t}.$$  

Therefore the bifurcation point is given by

$$\hat{\phi} = K_2 \left[ \frac{1 - e^{-r(T-t)}}{e^{-r(T-t)}(N(b_2) - 1) + 1 - N(b_1)} \right]. \quad (16)$$

The programs developed to determine the intercepts $\phi$ and $\theta$ in the bifurcated case were modified to search for the value of $K_1$ given $K_2$, $\sigma$, $r$, and $T - t$ such that $\hat{\phi} = \theta$. The values for $K_1$ and the corresponding bifurcation point $\hat{\phi}$ obtained via these programs and from equation 15 are shown in Table 7. Using these results the early exercise region was computed using the iterated integral method (IIM) to confirm that bifurcation did indeed occur.
Table 7: Bifurcation Validation Formula Comparison

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$K_1$ Formula</th>
<th>$K_1$ Code</th>
<th>IIM $K_1$</th>
<th>$\phi$ Formula</th>
<th>$\phi$ Code</th>
<th>IIM $\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7.6905</td>
<td>7.6875</td>
<td>7.6806</td>
<td>4.3095</td>
<td>4.3120</td>
<td>4.3194</td>
</tr>
<tr>
<td>3</td>
<td>4.0991</td>
<td>4.0969</td>
<td>4.0946</td>
<td>5.9009</td>
<td>5.9031</td>
<td>5.9054</td>
</tr>
<tr>
<td>4</td>
<td>7.0012</td>
<td>7.0000</td>
<td>7.0000</td>
<td>2.9988</td>
<td>3.0000</td>
<td>3.0000</td>
</tr>
<tr>
<td>5</td>
<td>5.3164</td>
<td>5.3150</td>
<td>5.3137</td>
<td>4.6836</td>
<td>4.6831</td>
<td>4.6863</td>
</tr>
</tbody>
</table>

Table 8: Option Parameters for Bifurcation Validation

<table>
<thead>
<tr>
<th>Number</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sigma = .4, r=0.1, T=1.0, K_2=10$</td>
</tr>
<tr>
<td>2</td>
<td>$\sigma = .4, r=0.1, T=1.0, K_2=12$</td>
</tr>
<tr>
<td>3</td>
<td>$\sigma = .4, r=0.2, T=1.0, K_2=10$</td>
</tr>
<tr>
<td>4</td>
<td>$\sigma = .5, r=0.1, T=1.0, K_2=10$</td>
</tr>
<tr>
<td>5</td>
<td>$\sigma = .4, r=0.1, T=2.0, K_2=10$</td>
</tr>
</tbody>
</table>

for these values. The corresponding values obtained from IIM are shown in Table 7 as well. The parameter values are contained in Table 8. They are numbered in each for direct correspondence.

Both the nonlinear code which sought the value of $K_1$ when $\phi = \theta$ and the analytic formula for $K_1$ yielded excellent results. The relative error between them is shown in Figure 8 for increasing values of $K_2$. Both of these techniques predict the point of bifurcation calculated by IIM accurately. This is further confirmation of the postulate that when $S_1 = 0$, the two asset option is equivalent to the one dimensional novel option of equation 10.

5 Further Higher Dimensional Implications

Here we present a generalization of the symmetric intersection point approximation and build some intuition for how the intersection point behaves in higher dimensions.

Using the finite difference Monte Carlo method the early exercise boundary for a three asset Bermudan put option with one early exercise date was obtained. Figure 9 shows this early exercise region. There are three early exercise surfaces. Although now surfaces as opposed to curves, the shape is reminiscent of the two asset boundary shape. Figure 10 depicts one of the early exercise boundaries from Figure 9.
As with the two asset case, there exists a wedge region in the three asset early exercise region that is now pyramidal in shape. Again the implications of this wedge region is no different than from that of the two asset case. There are asset triples that lie within the wedge region which have a higher intrinsic value than other points in the early exercise region where it is optimal to exercise and yet one does not exercise. The intuition is analogous to the two asset case. Taking a point within the wedge region, there is only one way that the intrinsic value of the option can decrease, which is for all three assets to increase in price, but there are now eight ways that the intrinsic value can increase. Again it is not that the immediate payoff in this region would be less than in other areas where exercise is optimal, merely that within the wedge region the likelihood that the time discounted payoff by waiting until the next exercise opportunity will be higher.

For the symmetric case shown, this wedge region terminates at an intersection point analogous to the two asset case. The implications of this point are identical. Prior to presenting the general \(d\) dimensional intersection point formula for the symmetric \(d\) asset case we develop some intuition about how the intersection point should behave as more asset are added.

Consider \(\psi_d\) to be the intersection point for the early exercise region of a \(d\) asset Bermudan option. We propose that

\[
\psi_d < \psi_{d-1} < \ldots < \psi_3 < \psi_2,
\]
and that
\[
\lim_{d \to \infty} \psi_d = 0.
\]

That is, the intersection point for a \(d\) asset option is closer to the origin than a \(d-1\) asset option. Also as the number of assets tends to infinity the intersection point tends to the origin.

The intuition is straightforward. Consider the intersection point of the three asset case. This point balances the likelihood of any beneficial volatility move in the underlying assets with interest obtained from immediate exercise. In the two asset world, neglecting drift, there is a three in four chance the intrinsic value of the option increases due to beneficial price movements in the underlying assets. With three assets it increases to a seven in eight chance, and in \(d\) assets to a \(2^d - 1\) in \(2^d\) chance. Thus as the number of assets increase not only does the likelihood of a beneficial price movement increase but as well the likelihood of a large beneficial price movement increases. For the case with an infinite number of underlying assets one is virtually guaranteed that one of the assets will benefit from a \(N(0, 1)\) draw that is negative and infinitely large, resulting in that asset to move from whatever value it has prior to this draw down to zero, where the intrinsic value of the option and its payoff can increase no further. Thus more assets increase the likelihood of significant downward moves, not just downward moves, making holding the option more valuable.

We can show this another way as well. Consider an option on \(d\) assets lying
at the intersection point. We denote this option by $V_d$ which corresponds to $V(S_1, S_2, \ldots, S_d)$. We know that

$$V_d > V_{d-1} > \ldots > V_2 > V_1 > V_0.$$  

It is clear that an option on many assets is more valuable than an option on fewer simply because there are more opportunities for beneficial price movements to occur.

In the symmetric case, the early exercise boundary and thus the intersection point is given by

$$K - \psi_d = V_d,$$

where $\psi_d$ is $S_*$ at the intersection point.

Thus we now consider some $i > j$ such that $V_i > V_j$. It follows from above that

$$K - \psi_i > V_j = K - \psi_j.$$  

Simplifying, we obtain
\[ \psi_i < \psi_j, \]

as expected.

Using the same approach as in the two asset case we derive a general formula for the intersection point of a symmetric \( n \) asset Bermudan put option. The intersection point formula is given by

\[ S \approx \frac{K r \sqrt{\pi \Delta T}}{\sigma} D_n, \quad (17) \]

where

\[ D_n = \frac{2^{n-1}}{n(n-1)(1 + \sqrt{\frac{2}{\pi}} A_n)}, \]

and where \( A_n \) is given by

\[ A_n = \int_{-\infty}^{\infty} e^{-2u^2} ((1 - \text{erf}(u))^{n-2} - 1) du. \]

In the two asset case \( A_2 = 0 \) and \( D_2 = 1 \) so equation (17) simplifies to equation (5) and in the case of three symmetric assets it simplifies to

\[ S \approx \frac{2}{3} \frac{K r \sqrt{\pi \Delta T}}{\sigma}. \quad (18) \]

Except for the two asset and three asset cases, we cannot obtain the solution to equation (17) in closed form. That said, the intersection point is easily obtained via numerical integration. Figure 11 shows the effect of the \( D_n \) term for increasing numbers of assets.

The intersection point is indeed a monotonically decreasing function of \( n \), as we argued above. The intersection point rapidly decreases for the first few assets in addition to the two asset case but becomes a situation of diminishing returns as more assets are added.

Previously we mentioned that the early exercise boundary tail values in the two asset case tend to the early exercise values for a single asset case. We can generalize this to higher dimensions as well. In the \( d \) asset case

\[ V(S_1, S_2, \ldots, S_d) \rightarrow V(S_2, \ldots, S_d) \quad \text{as} \quad S_1 \rightarrow \infty. \]
That is, as one asset tends to infinity then the option tends to the value of a $d-1$ asset option and the tails are given by

\[
\begin{align*}
K_2 - \phi(S_3, S_4, \ldots, S_d) &= V(\phi(S_3, S_4, \ldots, S_d), S_3, \ldots, S_d) \\
K_3 - \phi(S_2, S_4, \ldots, S_d) &= V(S_2, \phi(S_2, S_4, \ldots, S_d), \ldots, S_d) \\
& \vdots \\
K_d - \phi(S_2, S_3, \ldots, S_{d-1}) &= V(S_2, \ldots, S_{d-1}, \phi(S_2, S_3, \ldots, S_{d-1}))
\end{align*}
\] (19)

As well it follows from above that if $d - 1$ assets tend to infinity then

\[
V(S_1, S_2, \ldots, S_d) \to V(S_d) \quad \text{as} \quad S_1, S_2, \ldots, S_{d-1} \to \infty.
\]

The corresponding early exercise point is given by the following in whichever asset has not tended to infinity,

\[
K_d - S_d = V(S_d).
\]

Figure 12 shows a comparison of a single symmetric three asset put tail from Figure 9 with a single two asset early exercise boundary for identical
parameters. The two asset boundary is obtained from the iterated integral method and as well from the finite difference Monte Carlo method for 41 and 1000 asset steps. The three asset tail was calculated using the FDMC method using 41 and 101 steps. Figure 12 further confirms the dimension reduction of the exercise boundaries as one of the assets tends to infinity.

It can also be noted from Figure 12 that as 2 of the assets tend to infinity, in the three asset case, the tail value tends to the single asset early exercise point, which was confirmed. The value of the early exercise surface for the three asset case using 101 asset steps when two assets tend to infinity is $8.464. Using the validation technique outlines above gives a true result of $8.434 which represents a relative error of less than one percent.

![Figure 12: Tail Comparison of Three Asset to Two Asset Bermudan Option](image)

6 Two Asset Call Results

In their paper in 1998, Ibanez and Zapatero published a method for solving multi-asset options problems. In this paper they presented results for two and three asset call options on continuous dividend paying assets. To compare the results from our finite difference Monte Carlo method (FDMC: see appendix B) with those obtained in [8] Bermudan call options on two and three assets with nine exercise dates were solved.

For call options with dividends the GBM stock process is given by
\[ dS = (r - \delta)Sdt + \sigma \sqrt{dt}dZ, \]

and the payoff function is

\[ H_k(S_i, S_j, t) = \max(\max(S_i - K, S_j - K), 0). \]

The parameters shown in Table 9 were used for both the two and three asset call options problems presented here.

**Table 9: Two Asset Bermudan Call Option Parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r \ (yrs^{-1}) )</td>
<td>0.05</td>
</tr>
<tr>
<td>( \sigma \ (yrs^{-\frac{1}{2}}) )</td>
<td>0.2</td>
</tr>
<tr>
<td>( \delta \ (yrs^{-1}) )</td>
<td>0.1</td>
</tr>
<tr>
<td>( T \ (yrs) )</td>
<td>3</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0.0</td>
</tr>
<tr>
<td>( K \ ($)</td>
<td>100</td>
</tr>
</tbody>
</table>

Figure 13 shows the early exercise region for a Bermudan call option on two continuous dividend paying underlying assets and five exercise dates. This was priced with a dividend value of 0.18. Graphically this appears quite similar to the result published by Ibanez and Zapatero[8] for the same parameters.

Table 10 and 11 presents the results obtained for two asset call. Forty one and one hundred one asset steps were used respectively in each asset direction for varying numbers of Monte Carlo draws. The results are in reasonably good agreement with the “true” results (binomial), with the largest relative error obtained in the option values being less than two percent for the forty one asset step case.

**Table 10: Bermudan Call Option on Two Assets with 41 Asset Steps: Comparison Against Number of Monte Carlo Draws**

<table>
<thead>
<tr>
<th>( S_0 )</th>
<th>5000</th>
<th>7500</th>
<th>10000</th>
<th>I&amp;Z</th>
<th>Binomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>8.1096</td>
<td>8.1939</td>
<td>8.0887</td>
<td>8.0729</td>
<td>8.075</td>
</tr>
</tbody>
</table>
There are three main things which can affect the accuracy of the results. They are: the number of Monte Carlo draws utilized, the number of asset steps taken in each asset and the order of the interpolant employed. With increasing number of Monte Carlo draws—all things being equal—it is expected that the accuracy will increase and the option value will converge to the true value. This should also be true for increasing numbers of asset steps and is coupled with the order of the interpolant.

As well, a Bermudan call option problem with nine exercise dates on three underlying dividend paying assets was solved. Forty one asset steps were used in each asset direction. The results are shown in Table 12 for various number of Monte Carlo draws.

Again the results are reasonably close to the true values and those obtained by Ibanez and Zapatero. The results are also likely affected by the same conditions that affected the two asset call option. Indeed this would also be true for put option cases as well.
Table 12: Bermudan Call Option on Three Assets: Comparison Against Number of Monte Carlo Draws

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>5000</th>
<th>50000</th>
<th>I&amp;Z</th>
<th>Binomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>11.0209</td>
<td>11.3718</td>
<td>11.2635</td>
<td>11.29</td>
</tr>
<tr>
<td>100</td>
<td>18.3169</td>
<td>18.7100</td>
<td>18.6716</td>
<td>18.69</td>
</tr>
<tr>
<td>110</td>
<td>27.0757</td>
<td>27.4795</td>
<td>27.5487</td>
<td>27.58</td>
</tr>
</tbody>
</table>

Figure 14 shows an early surface obtained for the last early exercise date. The region above the surface is the continuation (hold) region and the area lying below the surface is the exercise region. This is only with respect to two of the assets. There are two additional exercise surfaces for this exercise date. For clarity they are not shown since they would obscure the view of each other.

Figure 14: Three Asset Bermudan Call Option Exercise Boundary

The shape is generally consistent with expectations except for large values of the stock prices. The surface should continue linearly with no additional curvature other than that located around the strike price. While truncating the put option after a certain number of standard deviations is sufficient – since the option value decays to zero – it may not be for the call option where with increasing asset values, the call option increases indefinitely. Thus for large values of the asset price this asymptotic approximation is likely undervaluing the option in this region which would explain the observed
exercise boundary behaviour.

7 Discussion

In this paper we present novel validation formulae which are explicitly based on Bermudan options. Many published pricing techniques make the Bermudan approximation to American options. The Bermudan results sometimes differ from corresponding multi-dimensional American results and so our validation techniques can be used to validate Bellman’s principle based American options solvers.
A Multi-Dimensional Intersection Point

For a put option paying no dividends on the max of n symmetric assets
\((\max (\max (K - S_1, K - S_2, \ldots, K - S_n), 0)), \) call \(S\) the smallest asset value
beneath which it is always optimal to exercise the option. At this value, if
the option is exercised, then the interest paid on the early exercise profit
between the interval \(t\) and \(t + 1\) is

\[
(K - S)(e^{r\Delta T} - 1).
\]  

Similarly, if the option is held, this certain profit is forgone in favour of the
possibility that either stock falls below \(S\). This expected forgone profit is

\[
E[\max (\max (K - S_1, K - S_2, \ldots, K - S_n), 0)] - (K - S).
\]  

Now, assuming that \(Se^{r\Delta T} << K\) then

\[
E[\max (\max (K - S_1, K - S_2, \ldots, K - S_n), 0)] \approx E[\max (K - S_1, K - S_2, \ldots, K - S_n)],
\]

and

\[
E[\max (K - S_1, K - S_2, \ldots, K - S_n)] = K - E[\min (S_1, S_2, \ldots, S_n)].
\]  

Thus the expected forgone profit is approximately

\[
S - E[\min (S_1, S_2, \ldots, S_n)].
\]  

The certain forgone profit is balanced against the interest paid on the early
exercise, which results in

\[
(K - S)(e^{r\Delta T} - 1) = S - E[\min (S_1, S_2, \ldots, S_n; \Delta T), S].
\]  

Now, if \(S_1, S_2 \) to \(S_n\) are drawn from the same pdf \(f\) and \(y = \min (S_1, S_2, \ldots, S_n)\) then

\[
y \in nf(y)[1 - F(y)]^{n-1},
\]

where \(F(y)\) is the cumulative distribution of \(f\). Thus the expected value of
\(y\) is
\[ \int_{0}^{\infty} nyf(y)[1 - F(y)]^{n-1}dy. \] (27)

Using the probability density function given by the Green’s function for the Black-Scholes equation we have

\[ f(S_1) = \frac{1}{S_1\sigma\sqrt{2\pi\Delta T}} e^{-\frac{(\ln(S_1) - (r - \frac{1}{2}\sigma^2)\Delta T)^2}{2\sigma^2\Delta T}}, \] (28)

and

\[ F(S_1) = \frac{1}{\sigma\sqrt{2\pi\Delta T}} \int_{0}^{S_1} \frac{1}{S} e^{-\frac{(\ln(S) - (r - \frac{1}{2}\sigma^2)\Delta T)^2}{2\sigma^2\Delta T}} dS, \] (29)

which can be integrated and rearranged to obtain

\[ 1 - F(S_1) = \frac{1}{2} \left( 1 - \text{erf} \left( \frac{\ln(S_1) - (r - \frac{1}{2}\sigma^2)\Delta T}{\sigma\sqrt{2\Delta T}} \right) \right). \] (30)

Using the following change of variables

\[ \gamma = \frac{\ln(S_1) - (r - \frac{1}{2}\sigma^2)\Delta T}{\sigma\sqrt{2\Delta T}}, \] (31)

the expected value \( E[\min (S_1, S_2, \ldots, S_n)] \) becomes

\[ E[\min (S_1, S_2, \ldots, S_n)] = \frac{S_1 \exp((r-\frac{1}{2}\sigma^2)\Delta T)}{\sqrt{\pi^{2n-1}}} \int_{-\infty}^{\infty} ne^{\frac{\gamma^2}{2\Delta T}} \gamma e^{-\gamma^2} (1 - \text{erf} (\gamma))^{n-1}d\gamma. \] (32)

A further change of variables

\[ u = \gamma - \frac{\sigma\sqrt{\Delta T}}{\sqrt{2}}, \] (33)

gives

\[ E[\min (S_1, S_2, \ldots, S_n)] = \frac{nS_1 e^{r\Delta T}}{2^{n-1}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} (1 - \text{erf} (u + x))^{n-1}du, \] (34)

where
We now approximate this as follows. Let

\[ G(x) = \int_{-\infty}^{\infty} e^{-u^2} (1 - \text{erf}(u + x))^{n-1} du, \text{ } x \text{ small}, \]  

(36)

and expand in a Taylor series about \( x = 0 \). Evaluating the coefficients of the Taylor series and using equation 35 yields

\[ G(x) = \frac{2^n}{2n} \sqrt{\frac{\pi}{n}} - \frac{2(n-1)\sigma \sqrt{\Delta T}}{\sqrt{2\pi n}} \int_{-\infty}^{\infty} e^{-2u^2}(1 - \text{erf}(u))^{n-2} du. \]  

(37)

Substituting 34 into 25 and expanding both sides of the equation 25 in \( \Delta T \) and keeping terms upto \( O(\Delta T) \) we obtain

\[ (K - S)(r\Delta T) = S[-r\Delta T + \frac{2n(n-1)\sigma \sqrt{\Delta T}}{2n-1}\sqrt{\frac{\pi}{n}} \int_{-\infty}^{\infty} e^{-2u^2}(1 - \text{erf}(u))^{n-2} du.] \]  

(38)

Let

\[ I_n = \int_{-\infty}^{\infty} e^{-2u^2}(1 - \text{erf}(u))^{n-2} du. \]  

(39)

Simplifying we obtain

\[ \frac{Kr\sqrt{\Delta T}}{2n(n-1)\sigma} 2^{n-1} = \frac{SI_n}{\pi \sqrt{2}}, \]  

(40)

which we rearrange to

\[ S \approx \frac{Kr\sqrt{\pi \Delta T}}{\sigma} D_n, \]  

(41)

where

\[ D_n = \frac{2^{n-1}}{n(n-1)(1 + \sqrt{\frac{2}{\pi} A_n})}, \]  

(42)

and where \( A_n \) is given by
\[ A_n = \int_{-\infty}^{\infty} e^{-2u^2}((1 - \text{erf}(u))^{n-2} - 1)du. \]  

(43)

We can obtain analytic solutions for the two asset (n=2) and three asset (n=3) cases. For the two asset case we obtain

\[ S \approx \frac{Kr \sqrt{\pi \Delta T}}{\sigma}, \]  

(44)

and for \( n = 3 \) we obtain

\[ S \approx \frac{2}{3} \frac{Kr \sqrt{\pi \Delta T}}{\sigma}. \]  

(45)

For \( n > 3 \) an analytic form for \( A_n \) remains elusive. Intersection point values for \( n > 3 \) can be easily obtained numerically.
B Finite Difference (Lattice) Monte Carlo Method

Continuing in a similar manner to the iterated method and again utilizing dynamic programming methodology, our second technique involves solving for the multi-asset option via Monte Carlo simulation in an effort to avoid some of the limitations that arose in the previous technique. For simplicity, we refer again to a two dimensional case, though the following procedure is analogous in every way for the higher dimensional cases.

We start by discretizing each exercise surface of the multi-asset Bermudan option into a uniform lattice of stock d-tuples. This is essentially a finite difference lattice hence the name finite difference Monte Carlo (FDMC). As can be seen in Figure 15, each exercise surface has an associated d-dimensional lattice where each node of the lattice is a corresponding stock d-tuple \((S_1, S_2, ..., S_d)\).

![Figure 15: FDMC Diagram Depicting Two Dimensional Lattices at Each Exercise Opportunity](image)

However, instead of solving for the option value at each surface by making a finite difference approximation of the partial differential equation and working backwards, we utilize Monte Carlo simulation to calculate the option
value at each asset pair (d-tuple) \((S_1, S_2)\) on each exercise surface starting from contract expiry and working backward in time to the desired starting stock pair. At each early exercise surface one is essentially solving a multi-dimensional integral dependant upon the \textit{a posteri} surface.

At contract expiry the value \(V(S_1, S_2)\) for any asset pair \((S_1, S_2)\) on the lattice is known. It is simply calculated from the payoff function \(H\). Clearly at expiry the option either has some payoff value or it is worthless. To determine the value of the option at any other asset pair \((S_1, S_2)\) on any other exercise surface excluding expiry we work backward inductively solving

\[
V_{i-1}(S_1, S_2) = e^{-r(T-t)}E^Q[H(S_1, S_2, V_i(S_1, S_2))]
\]

where

\[
V_N = 0.
\]

When we have completed recursively calculating to \(V_1(S_1, S_2)\) for all points \((S_1, S_2)\) on the “final” (actually first) early exercise surface lattice \(i = 1\) we can then solve for the option price at the desired starting stock price pair – or any stock price pair – \(V_0(S_1, S_2)\).

\textbf{B.0.1 Two Dimensional Case}

Given a Bermudan put option with \(N\) early exercise opportunities, expiring at \(T\) on two underlying assets \(S_1, S_2\) (stocks) with correlation \(\rho\), volatilities \(\sigma_1, \sigma_2\) respectively and a risk free interest rate \(r\), start by partitioning the option state space into \(N + 1\) exercise surfaces which correspond to the \(N\) early exercise opportunities and expiry, separated by a time increment \(\Delta T\), where

\[
\Delta T = \frac{T - t}{N + 1}.
\]

At each exercise opportunity, a two dimensional lattice is constructed with spatial step sizes \(\text{step}_1, \text{step}_2\) in each asset \(S_1, S_2\) respectively. The number of nodes (steps) in each spatial (asset) direction must remain finite however so one must employ a criterion for limiting the lattice size. For this we need an understanding of how the option value behaves asymptotically as either/both asset(s) tends to infinity.

We rely upon an intuitive argument to that end. An option is related to the probability that the underlying assets expire in the money. The payoff is dependant on where the option expires as well. To build intuition we start
with the simplest one dimensional put option. In the case of a put option on a single underlying stock we can look at what happens when the underlying stock is far out of the money. Neglecting drift, in any one price move the expected change is the stock price is given by

\[ |dS| = S\sigma \sqrt{\Delta T} N(0, 1), \]

equally likely gaining (upward) or losing (downward) value. Neglecting drift is not a strong assumption; in any small time move \( dt \) the volatility term dominates the drift term since volatility scales with \( \sqrt{dt} \).

In order for the option not to expire worthless we must have a series of \( S\sigma \sqrt{\Delta T} N(0, 1) \) downward moves or a single large downward move requiring a large negative \( N(0, 1) \) draw. Either situation is unlikely to occur and thus the option is most likely to expire worthless. Going further out of the money from this point one would expect there to be minimal change to the option value simply because we have already determined that the option is worthless. How much more worthless can it get? Thus for large stock prices (in the case of puts, opposite for calls) we expect the slope of the option value to be essentially flat. Furthermore we expected it to remain flat even as we move closer to the strike price until the point where we are within a region close enough to the strike price such that the likelihood of a move which places the stock in the money is significant. That is, we can measure the distance the stock price is from the strike in terms of standard deviations of the stock price. Based on this reasoning, the state space is truncated after six standard deviations from the strike price such that each stock exists on the discrete sample

\[ S_i \in [0, 6K\sigma \sqrt{T}] \]

with \( I_i \) steps of size \( \text{step}_i \).

For each stock pair \((S_i, S_j)\) on a given lattice \( k \) we simulate forward in time onto the \((k+1)^{st}\) surface using \( M \) Monte Carlo simulants. The value at any stock pair is given by

\[ V_k(S_i, S_j) = \frac{1}{M} \sum_{m=1}^{M} H_k(S_i^m, S_j^m), \quad (46) \]

where the payoff function \( H \) is given by

\[ H_k(S_i, S_j) = \left\{ \begin{array}{ll}
\max \left( \max (K - S_i, K - S_j), 0 \right) & : k = N \\
\max \left( \max (K - S_i, K - S_j), V_{k+1}(S_i, S_j) \right) & : k = N - 1..1.
\end{array} \right. \quad (47) \]
B.0.2 Monte Carlo Simulated Asset Prices

The Monte Carlo simulators evolve the asset’s price forward in time from exercise opportunity $k$ to $k + 1$ according to the stochastic process that the underlying assets follow. Depending upon the nature of this process, any single Monte Carlo simulant between any exercise opportunity $k$ and $k + 1$, may need to be itself simulated forward multiple times. If the stochastic process is integrable then each Monte Carlo simulant is given by its integrated form. If the process cannot be obtained in closed form, then it must be integrated numerically[10]. In this case the Monte Carlo simulant is the final value obtained from the numerical integration between $k$ and $k + 1$.

In the case of integratable GBM stock processes, the simulated asset prices for each stock $j$ follow

$$S_{i+1}^j = S_i e^{(r + \frac{1}{2} \sigma_j^2)dt + \sigma_j \sqrt{t} \Phi(j)}, \quad (48)$$

where $\Phi(j), j = 1..d$ are normal $N(0, 1)$ random variable with correlation $\rho$.

We obtain the correlated random variables through a two step process. First a set of $M$ uncorrelated normal $N(0, 1)$ random variables are generated for each stock. The normal random variates are created from uniform random variates using the Box-Mueller Method. Next the sets of uncorrelated normal random variables $\varepsilon_j$ are correlated using Cholesky decomposition.[15]

In the multi-dimensional setting the correlation of stocks with one another must be considered in the pricing of options. The correlation is described by a correlation matrix $\Sigma$ where

$$\Sigma = \rho_{ij}. \quad (49)$$

Each entry $\rho_{ij}$ in the $d \times d$ correlation matrix $\Sigma$ relates any one stock $i$ to any other stock $j$ among the $d$ stocks. The correlation between any two stocks $S_i, S_j$ is given by the corresponding entry $\rho_{ij}$ in the correlation matrix.

We utilize this correlation matrix $\Sigma$ and Cholesky decomposition to correlate the sets of uncorrelated normal random variates $\varepsilon_j$ with each other. Given the matrix $\Sigma$ we wish to obtain via Cholesky decomposition a matrix $M$ such that

$$\Phi = M \varepsilon,$$

where $\Phi$ is a matrix of correlated random variates, $\varepsilon$ a matrix of uncorrelated random variates, and the columns of each matrix represent the correlated/uncorrelated random variates respectively for each asset.
References


