

# Algebraic properties of the Lambert $W$ Function from a result of Rosenlicht and of Liouville

Manuel Bronstein\*, Robert M. Corless<sup>1</sup>,  
James H. Davenport<sup>2</sup>, and D.J. Jeffrey<sup>1</sup>

<sup>1</sup> Ontario Research Centre for Computer Algebra and the  
Department of Applied Mathematics,  
University of Western Ontario,  
London, CANADA, N6A 5B7  
{rcorless,djeffrey}@uwo.ca

<sup>2</sup>Department of Mathematical Sciences  
University of Bath  
Bath BA2 7AY  
England  
jhd@maths.bath.ac.uk

## Abstract

It is shown that the Lambert  $W$  function cannot be expressed in terms of the elementary, Liouvillian, functions. The proof is based on a theorem due to Rosenlicht. A related function, the Wright  $\omega$  function is similarly shown to be not Liouvillian.

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The Lambert  $W$  function [5, 9] is a multi-valued function defined as the solution of

$$W(x)e^{W(x)} = x, \quad (1)$$

one of the simplest possible non-algebraic equations. The Wright  $\omega$  function [4] also satisfies a simple transcendental equation (away from its discontinuities):

$$\omega(x) + \ln \omega(x) = x. \quad (2)$$

Both of these functions are implicitly elementary, in the sense discussed by Risch in [7]. One can ask whether there are explicit formulations of those functions

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\*This paper is dedicated to the memory of Manuel Bronstein (1963–2005).

in terms of known functions, or whether they are genuinely new functions. A common class of “well-known” functions are the Liouvillian functions.

**Definition 1** *Let  $(k, ')$  be a differential field of characteristic 0. A differential extension  $(K, ')$  of  $k$  is called Liouvillian over  $k$  if there are  $\theta_1, \dots, \theta_n \in K$  such that  $K = C(x, \theta_1, \dots, \theta_n)$  and for all  $i$ , at least one of the following holds:*

1.  $\theta_i$  is algebraic over  $k(\theta_1, \dots, \theta_{i-1})$ ;
2.  $\theta_i' = \eta$  for some  $\eta \in k(\theta_1, \dots, \theta_{i-1})$ ;
3.  $\theta_i'/\theta_i = \eta$  for some  $\eta \in k(\theta_1, \dots, \theta_{i-1})$ .

*We say that  $f(x)$  is a Liouvillian function if it lies in some Liouvillian extension of  $(C(x), d/dx)$  for some constant field  $C$ .*

It turns out that the possible closed-form expressions for solutions of equations of the form (1–2) were already studied by Liouville [6], who was certainly able to prove already that  $W(x)$  is not a Liouvillian function. In any event, this result was known to Rosenlicht, who published in [8] a proposition that can be applied to prove easily that  $W(x)$  and  $\omega(x)$  (or many functions defined by similar transcendental equations) are not Liouvillian. Yet, questions about whether  $W(x)$  is elementary or Liouvillian appear in the literature [3], possibly because Rosenlicht’s paper is not as well-read as it deserves to be, so we illustrate in this note how Rosenlicht’s theorem can prove that neither  $W(x)$  nor  $\omega(x)$  are Liouvillian.

We start by recalling Rosenlicht’s result.

**Proposition 1** [8, Proposition, p.21] *Let  $k$  be a differential field of characteristic zero and let  $y_1, \dots, y_n, z_1, \dots, z_n$  be elements of a Liouvillian extension of  $k$  having the same subfield of constants as  $k$ . Suppose that*

$$\frac{y_i'}{y_i} = z_i', \quad i = 1, \dots, n,$$

*and that  $k(y_1, \dots, y_n, z_1, \dots, z_n)$  is algebraic over each of its subfields  $k(y_1, \dots, y_n)$  and  $k(z_1, \dots, z_n)$ . Then,  $y_1, \dots, y_n, z_1, \dots, z_n$  are all algebraic over  $k$ .*

An immediate consequence of the case  $n = 1$  of that proposition is that if  $W(x)$  and  $\omega(x)$  are Liouvillian functions, then they must be algebraic functions: suppose that  $W$  belongs to a Liouvillian extension  $K$  of  $\mathbb{C}(x)$ . Take  $k = C(x)$  where  $C$  is the constant subfield of  $K$ , then  $K$  is Liouvillian over  $k$  and both fields have the same subfield of constants. Taking logarithmic derivatives on both sides of (1) yields

$$W'/W + W' = 1/x, \tag{3}$$

whence  $y'/y = W'$  where  $y = x/W \in K$ . Since  $k(y, W) = k(y) = k(W)$ , Rosenlicht’s theorem implies that  $W$  is algebraic over  $k = C(x)$ . The proof is similar for  $\omega(x)$ : differentiating both sides of (2) yields  $\omega' + \omega'/\omega = 1$ , whence

$\omega'/\omega = z'$  where  $z = x - \omega$ . Since  $k(\omega, z) = k(\omega) = k(z)$ , Rosenlicht's theorem implies that  $\omega$  is algebraic over  $k = C(x)$ .

There are obvious analytic arguments why  $W(x)$  and  $\omega(x)$  cannot be algebraic functions, so they cannot be Liouvillian functions: if  $W(x)$  has a pole of finite order, then  $e^{W(x)}$ , and therefore  $W(x)e^{W(x)}$ , have an essential singularity, so  $W(x)e^{W(x)}$  cannot equal  $x$ . Similarly if  $\omega(x)$  has a zero, then  $\ln \omega(x)$ , and therefore  $\omega(x) + \ln \omega(x)$ , have a logarithmic singularity, so  $\omega(x) + \ln \omega(x)$  cannot equal  $x$ . Since algebraic functions with either no pole or no zero must be constants, and  $W(x)$  and  $\omega(x)$  cannot be constant, they cannot be algebraic.

The above argument can be cast in algebraic terms. Since Rosenlicht proved his result algebraically, we outline the algebraic proof that  $W(x)$  and  $\omega(x)$  cannot be algebraic functions. Note that (3) implies that  $y = W(x)$  is a solution of the differential equation

$$xy'(1+y) = y. \quad (4)$$

We first recall some notations and results from [2]: we say that a field  $E$  is an algebraic function field of one variable over a subfield  $F \subset E$  if

- $E$  is of transcendence degree 1 over  $F$ ,
- for any  $t \in E$  transcendental over  $F$ ,  $[E : F(t)]$  is finite.

By an  $F$ -place of  $E$ , we then mean the maximal ideal of a valuation ring of  $E$  containing  $F$ . For such a place  $p$ , we write  $\nu_p : E^* \rightarrow \mathbb{Z}$  for its order function. It has in particular the following properties:

- $\nu_p(c) = 0$  for any  $c \in \overline{F} \cap E^*$ .
- $\nu_p(ab) = \nu_p(a) + \nu_p(b)$  and  $\nu_p(a+b) \geq \min(\nu_p(a), \nu_p(b))$  for any  $a, b \in E^*$ .
- $\nu_p(a+b) = \min(\nu_p(a), \nu_p(b))$  for any  $a, b \in E^*$  such that  $\nu_p(a) \neq \nu_p(b)$ .
- For any  $a \in E^*$ , if  $\nu_p(a) \geq 0$  at all the  $F$ -places of  $E$ , then  $a$  is algebraic over  $F$ .

Let now  $t \in E$  be transcendental over  $F$  and  $p$  be any  $F$ -place of  $E$ . We write  $r_t(p) \in \mathbb{Z}_{>0}$  for the ramification index of  $p$  over  $F(t)$ . In addition, we call the place  $p$  *infinite* (w.r.t.  $t$ ) if  $t^{-1} \in p$ , *finite* (w.r.t.  $t$ ) otherwise. A finite place  $p$  contains a unique monic irreducible  $P \in F[t]$ , called the *center of  $p$*  (w.r.t.  $t$ ).

**Proposition 2** *Let  $(F, ')$  be a differential field containing an element  $x$  such that  $x' = 1$ . If  $F$  has transcendence degree 1 over its constant subfield, then the only solution  $y \in F$  of (4) is  $y = 0$ .*

**Proof.** Let  $C$  be the constant subfield of  $F$  and suppose that  $F$  has transcendence degree 1 over  $C$ . Since  $x' = 1$ ,  $x$  is transcendental over  $C$ , so  $F$  is algebraic over  $C(x)$ . Let  $y \in F$  be a nonzero solution of (4) and  $E = \overline{C}(x, y)$ , which is an algebraic function field of one variable over  $\overline{C}$ . Let  $p$  be any  $\overline{C}$ -place of  $E$ . Applying  $\nu_p$  on both sides of (4), we get

$$\nu_p(x) + \nu_p(y') + \nu_p(1+y) = \nu_p(y). \quad (5)$$

Suppose that  $\nu_p(y) < 0$ . Then,  $\nu_p(1 + y) = \min(0, \nu_p(y)) = \nu_p(y)$  and (5) becomes

$$\nu_p(x) + \nu_p(y') = 0. \quad (6)$$

If  $p$  is finite w.r.t.  $x$ , then  $\nu_p(x) \geq r_x(p)$ . But Lemma 1.7 of [1] implies that  $\nu_p(y') = \nu_p(y) - r_x(p) < -r_x(p)$ , in contradiction with (6). If  $p$  is infinite, then  $\nu_p(x) = -r_x(p)$ . But Lemma 1.8 of [1] implies that  $\nu_p(y') \leq \nu_p(y) + r_x(p) < r_x(p)$ , in contradiction with (6). Therefore  $\nu_p(y) \geq 0$  at all the  $\overline{C}$ -places of  $E$ , which implies that  $y \in \overline{C}$ , hence that  $y' = 0$ , and (4) becomes  $0 = y$ .  $\square$

Since the only algebraic solution of (4) is 0, which is not a solution of (1),  $W(x)$  cannot be algebraic, hence it cannot be a Liouvillian function.

The proof that  $\omega(x)$  is not an algebraic function is similar, since  $y = \omega(x)$  is a solution of the differential equation  $y'(1 + y) = y$ . The equalities (5) and (6) become respectively  $\nu_p(y') + \nu_p(1 + y) = \nu_p(y)$  and  $\nu_p(y') = 0$ , and the proof of Proposition 2 remains valid.

## References

- [1] M. Bronstein. Integration of Elementary Functions. *J. Symbolic Computation*, **9**(2):117–174, 1990.
- [2] C. Chevalley. *Algebraic Functions of One Variable*. American Mathematical Society, New York, 1951.
- [3] Timothy Chow. What is a closed-form number? *American Mathematical Monthly*, **106**(5): 440–448, 1999.
- [4] R.M. Corless & D.J. Jeffrey, The Wright  $\omega$  function. In: Artificial Intelligence, Automated Reasoning, and Symbolic Computation. (Editors: J. Calmet, B. Benhamou, O. Caprotti, L. Henocque, V. Sorge). AISC-Calculamus 2002. LNAI 2385, Springer. pp 76–89, 2002.
- [5] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey & D.E. Knuth. On the Lambert  $W$  function. *Advances in Computational Mathematics* **5**: 329–359, 1996.
- [6] J. Liouville. Mémoire sur la classification des transcendentes et sur l'impossibilité d'exprimer les racines de certaines équations en fonction finie explicite des coefficients. *Journal de mathématiques pures et appliquées* 2, 3: pp. 56–105, 523–547, 1837 and 1838.
- [7] Robert H. Risch. Implicitly Elementary Integrals. *Proceedings of the American Mathematical Society* **57**(1): 1–7, 1976.
- [8] Maxwell Rosenlicht, On the explicit solvability of certain transcendental equations, *Inst. Haute Etudes Sci. Publ. Math.* **36**, 15–22, 1969.
- [9] E.M. Wright. Solution of the equation  $ze^z = a$ . *Bull. Amer. Math. Soc.* **65**: 89–93, 1959.