Transcendental equations satisfied by the individual zeros of Riemann ζ, Dirichlet and modular L-functions

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Abstract

We consider the non-trivial zeros of the Riemann ζ-function and two infinite classes of L-functions; Dirichlet L-functions and those based on level one modular forms. For these functions, we show that there are an infinite number of zeros on the critical line in one-to-one correspondence with the zeros of the cosine function, and thus enumerated by an integer n. From this it follows that the ordinate of the n-th zero satisfies a transcendental equation that depends only on n. Under weak assumptions, we show that the number of solutions of this equation already saturates the counting formula on the entire critical strip, constituting therefore a concrete proposal toward verifying the Riemann hypothesis for these classes of functions. We compute numerical solutions of these transcendental equations and also its asymptotic limit of large ordinate. The starting point is an explicit formula, yielding an approximate solution for the ordinates of the zeros in terms of the Lambert W-function. This approach is a novel and simple method to numerically compute the non-trivial zeros of these functions. Employing these numerical solutions, in particular for the ζ-function, we verify that the leading order asymptotic expansion is accurate enough to confirm Montgomery's and Odlyzko's pair correlation conjectures and also to reconstruct the prime number counting function. Furthermore, the numerical solutions of the exact transcendental equation can determine the ordinates of the zeros to any desired accuracy. We also study in detail Dirichlet L-functions and the L-function for the modular form based on the Ramanujan τ-function, which is closely related to the bosonic string partition function.

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I. INTRODUCTION

Riemann's major contribution to number theory was an explicit formula for the arithmetic function $\pi(x)$, which counts the number of primes less than x, in terms of an infinite sum over the non-trivial zeros of the $\zeta(z)$ function, i.e. roots ρ of the equation $\zeta(z) = 0$ on the *critical strip* $0 \le \Re(z) \le 1$ [1]. It was later proven by Hadamard and de la Vallée Poussin that there are no zeros on the line $\Re(z) = 1$, which in turn proved the prime number theorem $\pi(x) \sim \mathrm{Li}(x)$. (See section VI C for a review.) Hardy proved that there are an infinite number of zeros on the *critical line* $\Re(z) = \frac{1}{2}$. The *Riemann hypothesis* (RH) was his statement, in 1859, that all zeros on the critical strip have $\Re(\rho) = \frac{1}{2}$. Despite strong numerical evidence of its validity, it remains unproven to this day. Many important mathematical results were proven assuming the RH, so it is a cornerstone of fundamental mathematics. Some excellent introductions to the RH are [2–4].

Riemann also gave an estimate $N(T)$, given by (14) but without the $\arg\zeta$ term, for the average number of zeros on the *entire critical strip* with $0 < \Im(\rho) < T$. This formula was later proven by von Mangoldt, but has it never been proven to be valid *on the critical line*, as explicitly stated in Edward's book [1]. Denoting the number of zeros on the critical line, up to height T, by $N_0(T)$, Hardy and Littlewood showed that $N_0(T) > C\,T$ and Selberg improved this result stating that $N_0(T) > C\,T\log T$ for very small C. Then, Levinson [6] demonstrated that $N_0(T) \ge CN(T)$ where $C = \frac{1}{3}$. This last result was further improved by Conrey [7] who obtained $C = \frac{2}{5}$. Obviously, if the RH is true then $N_0(T) = N(T)$. These statements are described in [1, Chap. 11] and [5, Chap. X].

Montgomery's conjecture that the non-trivial zeros satisfy the statistics of the eigenvalues of random hermitian matrices [8] led Berry to propose that the zeros are eigenvalues of a chaotic hamiltonian [9], along the lines of the original Hilbert-Polya idea. Further developments are in [10–14]. These works focus on $N(T)$ and carry out the analysis on the critical line, i.e. they essentially assume the validity of the RH. A number of interesting analytic results were obtained, emphasizing the important role of the function $S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + iT\right)$.

In a related, but essentially different approach by Connes based on adeles, there exists an operator playing the role of the hamiltonian, which has a continuous spectrum, and the Riemann zeros correspond to missing spectral lines [15]. We mention these interesting works because of the role of $N(T)$ in them, however, we will not be pursuing these ideas in this work. For interesting connections of the RH to physics see [16, 17] (and references therein).

L-functions are generalizations of the Riemann ζ-function, the latter being the trivial case [18]. In this paper we will consider two infinite classes of important L-functions, the Dirichlet L-functions and L-functions associated with modular forms. The former have applications primarily in multiplicative number theory, whereas the latter in additive number theory. These functions can be analytically continued to the entire (upper half) complex plane. The *generalized Riemann hypothesis* (GRH) is the conjecture that all non-trivial zeros of L-functions lie on the critical line. Much less is known about the zeros of L-functions in comparison with the ζ-function, however let us mention a few works. Selberg [19] obtained the analog of Riemann-von Mangoldt counting formula (14) for Dirichlet L-functions. Based on this result, Fujii [20] gave an estimate for the number of zeros on the critical strip with the ordinate between $[T+H, T]$. The distribution of low lying zeros of L-functions near and at the critical line was examined in [21], assuming the GRH. The statistics of the zeros, i.e. the analog of the Montgomery-Odlyzko conjecture, were studied in [22, 23]. It is also known that more than half of the non-trivial zeros of Dirichlet L-functions are on the critical line [24]. For a more detailed introduction to L-functions see [25].

Besides the Dirichlet L-functions, there are more general constructions of L-functions based on arithmetic and geometric objects, like varieties over number fields and modular forms [26, 27]. Some results for general L-functions are still conjectural. For instance, it is not even clear if some L-functions can be analytically continued into a meromorphic function. We will only consider the additional L-functions based on modular forms here. Thus the L-functions considered in this paper have similar properties, namely, they possess an Euler product, can be analytic continued into the (upper half) complex plane, except for possible poles at $z=0$ and $z=1$, and satisfy a non-trivial functional equation.

Since it is well known that there are an infinite number of zeros on the critical line for the Riemann ζ-function, if in some region of the critical strip one can show that the counting formula (14) correctly counts the zeros on the critical line, then this proves the RH in this region of the strip. Since it has been shown numerically that the first billion or so zeros all lie on the critical line [28, 29], one can approach this problem asymptotically. Such an analysis was carried out in [30] where the main outcome was an asymptotic transcendental equation for the ordinate of the n-th Riemann zero on the critical line. The way in which this equation is derived shows that these zeros are in one-to-one correspondence with the zeros of the cosine function; it is in this manner that the n-dependence arises. In this paper we provide a more rigorous and through analysis of this result. Moreover, we propose generalizations. We derive an *exact* equation satisfied by the Riemann zeros on the critical line, where the above mentioned asymptotic equation is obtained as a limit of large n. We also generalize these results to Dirichlet L-functions and to L-functions related to modular forms. For all these classes of functions we obtain an exact equation for the ordinate of the n-th zero on the critical line. Since such an equation comes from a relation with the cosine function, its solutions can be automatically counted. We will argue that, under weak assumptions, the number of solutions of the transcendental equation coincide with the counting formula for zeros on the entire critical strip, i.e. $N_0(T) = N(T)$.

We organize our work as follows. In section II we derive an exact equation satisfied by each individual Riemann zero on the critical line. We discuss how the number of its solutions can be the same as the counting formula on the whole strip. In section III we follow the same analysis for Dirichlet L-functions, and in section IV for L-functions based on level one modular forms. In section V we derive a useful approximation for the zeros expressed explicitly in terms of the Lambert W-function. In section VI we obtain numerical solutions to these equations for the Riemann ζ-function. We show that the leading order asymptotic approximation is accurate enough to reproduce the GUE statistics and the prime number counting function. Furthermore, we show that solutions to the exact transcendental equation yield highly accurate results, up to 500 digit accuracy or more if desired. In section VII we solve numerically the transcendental equation related to Dirichlet L-functions, considering two explicit examples. We also consider numerical solutions for L-functions based on modular forms, in particular for the L-function based on the Ramanujan τ-function. Section VIII contains our concluding remarks. In appendix A we unify the results proposed in this paper for a general, but not specific, class of L-functions. In appendix B we present the short Mathematica code we used to calculate the zeros.

II. ZEROS OF THE RIEMANN ζ-FUNCTION

For simplicity we first consider the Riemann ζ-function, which is the simplest Dirichlet L-function. Moreover, we first consider the asymptotic equation (12), first proposed in [30], since it involves more familiar functions. However, this asymptotic equation should be viewed as following trivially from the exact equation (19), presented later.

A. Asymptotic equation satisfied by the n-th zero

Let us start by defining the function

$$\chi(z) \equiv \pi^{-z/2}\,\Gamma\left(z/2\right)\zeta(z). \tag{1}$$

In quantum statistical physics, this function is the free energy of a gas of massless bosonic particles in d spatial dimensions when $z = d + 1$, up to the overall power of the temperature T^{d+1}. Under a "modular" transformation that exchanges one spatial coordinate with Euclidean time, if one analytically continues d, physical arguments [31] shows that it must have the symmetry

$$\chi\left(z\right) = \chi\left(1 - z\right). \tag{2}$$

This is the fundamental, and amazing, functional equation satisfied by the ζ-function, which was proven by Riemann. For several different ways of proving (2) see [5]. Now consider Stirling's approximation $\Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z}\left(1 + O\left(z^{-1}\right)\right)$, where $z = x + iy$, which is valid for large y. Under this condition we also have

$$z^z = \exp\left(i\left(y\log y + \frac{\pi x}{2}\right) + x\log y - \frac{\pi y}{2} + x + O\left(y^{-1}\right)\right). \tag{3}$$

Therefore, using the polar representation $\zeta = |\zeta| e^{i\arg\zeta}$ and the above expansions, we can write $\chi(z) = A\,e^{i\theta}$ where

$$A(x,y) = \sqrt{2\pi}\,\pi^{-x/2}\left(\frac{y}{2}\right)^{(x-1)/2} e^{-\pi y/4}|\zeta(x+iy)|\left(1 + O\left(z^{-1}\right)\right), \tag{4}$$

$$\theta(x,y) = \frac{y}{2}\log\left(\frac{y}{2\pi e}\right) + \frac{\pi}{4}(x-1) + \arg\zeta(x+iy) + O\left(y^{-1}\right). \tag{5}$$

The above approximation is very accurate. For y as low as 100, it evaluates $\chi\left(\frac{1}{2} + iy\right)$ correctly to one part in 10^6. Above we are assuming $y > 0$. The results for $y < 0$ follows trivially from the relation $\left(\chi(z)\right)^* = \chi(z^*)$.

Now let $\rho = x + iy$ be a Riemann zero. Then $\arg\zeta(\rho)$ can be well-defined by the limit

$$\arg\zeta\left(\rho\right) \equiv \lim_{\delta\to 0^{+}} \arg\zeta\left(x + \delta + iy\right). \tag{6}$$

For reasons that are explained below, it is important that $0 < \delta \ll 1$. This limit in general is not zero. For instance, for the first Riemann zero, $\arg\zeta\left(\frac{1}{2} + iy_1\right) \approx 0.1578739$. On the critical line $z = \frac{1}{2} + it$, if t does not correspond to the imaginary part of a zero, the well known function $S(t) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + it\right)$ is defined by continuous variation along the straight lines starting from 2, then up to $2 + it$ and finally to $\frac{1}{2} + it$, where $\arg\zeta(2) = 0$. Assuming the RH, the current best bound is given by $|S(t)| \leq \left(\frac{1}{2} + o(1)\right)\frac{\log t}{\log\log t}$ for $t \to \infty$, proven by Goldston and Gonek [32]. On a zero, the standard way to define this term is through the limit $S(\rho) = \frac{1}{2}\lim_{\epsilon\to 0}\left(S\left(\rho + i\epsilon\right) + S\left(\rho - i\epsilon\right)\right)$. We have checked numerically that for several zeros on the line, our definition (6) gives the same answer as this standard approach.

From (1) we have $\left(\chi(z)\right)^{*} = \chi\left(z^{*}\right)$, thus $A(x,-y) = A(x,y)$ and $\theta(x,-y) = -\theta(x,y)$. Denoting $\chi\left(1 - z\right) = A'\, e^{-i\theta'}$ we then have

$$A'(x,y) = A(1 - x, y), \qquad \theta'(x,y) = \theta(1 - x, y). \tag{7}$$

From (2) we also have $|\chi(z)| = |\chi(1 - z)|$, therefore $A(x,y) = A'(x,y)$ for *any* z on the critical strip.

Let us now consider approaching a zero $\rho = x + iy$ through the $\delta \to 0^{+}$ limit. From (1) it follows that $\zeta(z)$ and $\chi(z)$ have the same zeros on the critical strip, so it is enough to consider the zeros of $\chi(z)$. From (2) we see that if ρ is a zero so is $1 - \rho$. Then we clearly have [46]

$$\lim_{\delta\to 0^{+}}\left[\chi(\rho + \delta) + \chi(1 - \rho - \delta)\right] = \lim_{\delta\to 0^{+}} A(x + \delta, y)B(x + \delta, y) = 0, \tag{8}$$

where we have defined

$$B(x,y) \equiv e^{i\theta(x,y)} + e^{-i\theta'(x,y)}. \tag{9}$$

The second equality in (8) follows from $A = A'$. Then, in the limit $\delta \to 0^{+}$, a zero corresponds to $A = 0$, $B = 0$ or both. They can simultaneously be zero since they are not independent. If $B = 0$ then $A = 0$, since $A \propto |\zeta(z)|$. However, the converse is not necessarily true.

Since there is more structure in B, let us consider $B = 0$. The general solution of this equation is given by $\theta + \theta' = (2n + 1)\pi$, which are a family of curves $y(x)$. However, since $\chi(z)$ is analytic on the critical strip, we know that the zeros must be isolated points rather than curves, thus this general solution must be restricted. Let us choose the particular solution

$$\theta = \theta', \qquad \lim_{\delta\to 0^+} \cos\theta = 0. \tag{10}$$

On the critical line, the first equation of (10) is already satisfied. Now the second equation implies $\lim_{\delta\to 0^+}\theta\left(\frac{1}{2}+\delta, y\right) = \left(n+\frac{1}{2}\right)\pi$, for $n = 0, \pm 1, \pm 2, \ldots$, hence

$$n = \frac{y}{2\pi}\log\left(\frac{y}{2\pi e}\right) - \frac{5}{8} + \lim_{\delta\to 0^+}\frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+\delta+iy\right). \tag{11}$$

A closer inspection shows that the right hand side of (11) has a minimum in the interval $(-2,-1)$, thus n is bounded from below, i.e. $n \geq -1$. Establishing the *convention* that zeros are labeled by positive integers, $\rho_n = \frac{1}{2} + iy_n$ where $n = 1, 2, \ldots$, we must replace $n \to n-2$ in (11). Therefore, the imaginary parts of these zeros satisfy the transcendental equation

$$\frac{y_n}{2\pi}\log\left(\frac{y_n}{2\pi e}\right) + \lim_{\delta\to 0^+}\frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+\delta+iy_n\right) = n - \frac{11}{8} \qquad (n = 1, 2, \ldots). \tag{12}$$

In short, we have shown that, asymptotically, there are an infinite number of zeros on the critical line whose ordinates can be determined by solving (12). This equation determines the zeros on the upper half of the critical line. The zeros on the lower half are symmetrically distributed; if $\rho_n = \frac{1}{2} + iy_n$ is a zero, so is $\rho_n^* = \frac{1}{2} - iy_n$.

The left hand side of (12) is a monotonically increasing function of y, and the leading term is a smooth function. This is clear since the same terms appear in the staircase function $N(T)$ described below; see Remark 1. Possible discontinuities can only come from $\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iy\right)$, and in fact, it has a jump discontinuity by one whenever y corresponds to a zero on the line. However, if $\lim_{\delta\to 0^+}\arg\zeta\left(\frac{1}{2}+\delta+iy\right)$ is well defined for every y, then the left hand side of equation (12) is well defined for any y, and due to its monotonicity, there must be a unique solution for every n. Under this assumption, the number of solutions of equation (12), up to height T, is given by

$$N_0(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + \frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+iT\right) + O\left(T^{-1}\right). \tag{13}$$

This is so because the zeros are already numbered in (12), but the left hand side jumps by one at each zero, with values $-\frac{1}{2}$ to the left and $+\frac{1}{2}$ to the right of the zero. Thus we can replace $n \to N_0 + \frac{1}{2}$ and $y_n \to T$, such that the jumps correspond to integer values. In this way T will not correspond to the ordinate of a zero and δ can be eliminated.

Using Cauchy's argument principle one can derive the Riemann-von Mangoldt formula, which gives the number of zeros in the region $\{0 < x < 1, 0 < y < T\}$ inside the *critical strip*. This formula is given by [1, 5]

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + S(T) + O\left(T^{-1}\right). \tag{14}$$

where $S(T) = \frac{1}{\pi} \arg\zeta\left(\frac{1}{2} + iT\right)$. Note that it has the same form as the counting formula on the *critical line* that we have just found (13). Thus, under the assumptions we have described, we conclude that $N_0(T) = N(T)$ asymptotically. This means that our particular solution (10), leading to equation (12), already saturates the counting formula on the whole strip and there are no additional zeros from $A = 0$ in (8) nor from the general solution $\theta + \theta' = (2n+1)\pi$. This strongly suggests that (12) describes all non-trivial zeros of $\zeta(z)$, which must then lie on the critical line.

B. Exact equation for the n-th zero

Let us now repeat the above analysis but without considering an asymptotic expansion. The exact versions of (4) and (5) are

$$A(x,y) = \pi^{-x/2}|\Gamma\left(\tfrac{1}{2}(x+iy)\right)||\zeta(x+iy)|, \tag{15}$$

$$\theta(x,y) = \arg\Gamma\left(\tfrac{1}{2}(x+iy)\right) - \frac{y}{2}\log\pi + \arg\zeta(x+iy), \tag{16}$$

where again $\chi(z) = Ae^{i\theta}$ and $\chi(1-z) = A'e^{-i\theta'}$, with $A'(x,y) = A(1-x,y)$ and $\theta'(x,y) = \theta(1-x,y)$. The zeros on the critical line correspond to the particular solution $\theta = \theta'$ and $\lim_{\delta\to 0^+}\cos\theta = 0$. Thus $\lim_{\delta\to 0^+}\theta\left(\frac{1}{2}+\delta, y\right) = \left(n + \frac{1}{2}\right)\pi$ and replacing $n \to n-2$, the imaginary parts of these zeros must satisfy the exact equation

$$\arg\Gamma\left(\tfrac{1}{4} + \tfrac{i}{2}y_n\right) - y_n\log\sqrt{\pi} + \lim_{\delta\to 0^+}\arg\zeta\left(\tfrac{1}{2} + iy_n\right) = \left(n - \tfrac{3}{2}\right)\pi. \tag{17}$$

The Riemann-Siegel ϑ function is defined by

$$\vartheta(t) \equiv \arg\Gamma\left(\tfrac{1}{4} + \tfrac{i}{2}t\right) - t\log\sqrt{\pi}, \tag{18}$$

where the argument is defined such that this function is continuous and $\vartheta(0) = 0$. Therefore, there are an infinite number of zeros in the form $\rho_n = \frac{1}{2} + iy_n$, where $n = 1, 2, \ldots$, whose imaginary parts *exactly* satisfy the following equation:

$$\vartheta(y_n) + \lim_{\delta\to 0^+}\arg\zeta\left(\tfrac{1}{2} + \delta + iy_n\right) = \left(n - \tfrac{3}{2}\right)\pi \qquad (n = 1, 2, \ldots). \tag{19}$$

Expanding the Γ-function in (18) through Stirling's formula, one recovers the asymptotic equation (12).

Again, as discussed after (12), the first term in (19) is smooth and the whole left hand side is a monotonic increasing function. If $\lim_{\delta\to 0^+} \zeta\left(\frac{1}{2}+\delta+iy\right)$ is well defined for every y, then equation (19) must have a unique solution for every n. Under this condition it is valid to replace $y_n \to T$ and $n \to N_0 + \frac{1}{2}$, then the number of solutions of (19) is given by

$$N_0(T) = \frac{1}{\pi}\vartheta(T) + 1 + \frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+iT\right). \tag{20}$$

The exact Backlund counting formula, which gives the number of zeros on the critical strip with $0 < \Im(\rho) < T$, is given by [1, Chap. 6]

$$N(T) = \frac{1}{\pi}\vartheta(T) + 1 + S\left(T\right). \tag{21}$$

where $S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right)$. Therefore, comparing with the exact counting formula on the whole *critical strip* (21), we have $N_0(T) = N(T)$ exactly. This indicates that our particular solution, leading to equation (19), captures all the zeros on the strip, indicating that they should all be on the critical line.

In summary, if (19) has a unique solution for each n, then this saturates the counting formula for the entire critical strip and this would establish the validity of the RH.

C. Further remarks

Remark 1. The small shift by δ in (12) is essential since it smooths out $S(y) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iy\right)$, which is known to jump discontinuously at each zero. As is well known, $S(y)$ is a piecewise continuous function, but rapidly oscillates around zero with discontinuous jumps, as shown in FIG. 1a. However, when this term is added to the smooth part of $N(T)$, one obtains an accurate staircase function, which jumps by one at each zero on the line; see FIG. 1b. Note than $N(T)$ is necessarily a monotonically increasing function.

The reason δ needs to be positive in (19) is the following. Near a zero ρ_n, $\zeta(z) \approx (z-\rho_n)\,\zeta'(\rho_n) = (\delta + i\,(y-y_n))\,\zeta'(\rho_n)$. This gives $\arg\zeta(z) \approx \arctan\left((y-y_n)/\delta\right) + \text{const.}$. Thus, with $\delta > 0$, as one passes through a zero from below, $S(y)$ *increases* by one, as it should based on its role in the counting formula $N(T)$. On the other hand, if $\delta < 0$ then $S(y)$ would decrease by one instead, which cannot be the case.

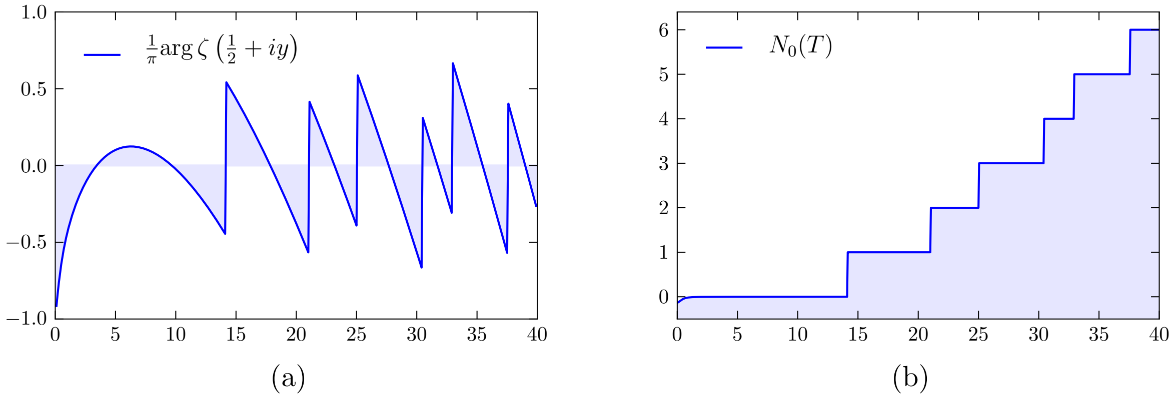

FIG. 1: (a) A plot of $\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iy\right)$ as a function of y showing its rapid oscillation. The jumps occur on a Riemann zero. (b) The function $N_0(T)$ in (13), which is indistinguishable from a manual counting of zeros.

Remark 2. An important consequence of equation (19), or its asymptotic version (12), is that all of these zeros are simple. This follows from the fact that they are in one-to-one correspondence with the zeros of the cosine function (10), which are simple. Conversely, if the zeros are simple, there is an easier way to see that the zeros correspond to $\cos\theta = 0$. On the critical line $z = \frac{1}{2} + iy$, the functional equation (2) implies $\chi(z)$ is real, thus for y not the ordinate of a zero, $\sin\theta = 0$ and $\cos\theta = \pm 1$. Thus $\cos\theta$ is a discontinuous function. Now let $y_\bullet$ be the ordinate of a *simple zero*. Then close to such a zero we define

$$c(y) \equiv \frac{\chi(\frac{1}{2}+iy)}{|\chi(\frac{1}{2}+iy)|} = \frac{y - y_\bullet}{|y - y_\bullet|}. \tag{22}$$

For $y > y_\bullet$ then $c(y) = 1$, and for $y < y_\bullet$ then $c(y) = -1$. Thus $c(y)$ is discontinuous precisely at a zero. In the above polar representation, formally $c(y) = \cos\theta(\frac{1}{2}, y)$. Therefore, by identifying zeros as the solutions to $\cos\theta = 0$, we are simply defining the value of the function $c(y)$ at the discontinuity as $c(y_\bullet) = 0$.

Remark 3. It is possible to introduce a new function $\zeta(z) \to \widetilde{\zeta}(z) = f(z)\zeta(z)$ that also satisfies the functional equation (2), i.e. $\widetilde{\chi}(z) = \widetilde{\chi}(1-z)$, but has zeros off of the critical line due to the zeros of $f(z)$. In such a case the corresponding functional equation will hold if and only if $f(z) = f(1-z)$ for any z, and this is a trivial condition on $f(z)$, which could have been canceled in the first place. Moreover, if $f(z)$ and $\zeta(z)$ have different zeros, the analog of equation (8) has a factor $f(z)$, i.e. $\widetilde{\chi}(\rho+\delta) + \widetilde{\chi}(1-\rho-\delta) = f(\rho+\delta)\left[\chi(\rho+\delta) + \chi(1-\rho-\delta)\right] = 0$, implying (8) again where $\chi(z)$ is the original (1). Therefore, the previous analysis eliminates $f(z)$ automatically and only finds the zeros of $\chi(z)$. The analysis is non-trivial precisely because $\zeta(z)$ satisfies the functional equation but $\zeta(z) \neq \zeta(1-z)$. Furthermore, it is a well known theorem that the only function which satisfies the functional equation (2) and has the same characteristics of $\zeta(z)$, is $\zeta(z)$ itself. In other words, if $\widetilde{\zeta}(z)$ is required to have the same properties of $\zeta(z)$, then $\widetilde{\zeta}(z) = C\,\zeta(z)$, where C is a constant [5, pg. 31].

Remark 4. Although equations (19) and (21) have an obvious resemblance, it is impossible to derive the former from the later, since the later is just a counting formula valid on the entire strip, and it is assumed that T is *not* the ordinate of a zero. Moreover, this would require the assumption of the validity of the RH, contrary to our approach, where we derived equations (19) and (12) on the critical line, without assuming the RH. Despite our best efforts, we were not able to find equations (12) and (19) in the literature. Furthermore, the counting formulas (13) and (21) have never been proven to be valid on the critical line [1].

Remark 5. One may object that our basic equations (12) and (19) involve $\zeta(z)$ itself and this is somehow circular. This is not a valid counter-argument. First of all, $\arg\zeta$ already appears in the counting function $N(T)$. Secondly, the equation (12) is a much more detailed equation than simply $\zeta(z) = 0$, which has an infinite number of solutions, in contrast with (12) and (19) which for each n have a unique solution corresponding to the n-th zero. Also, there are well-known ways to calculate $\arg\zeta$, for example from an integral representation or a convergent series [34].

III. ZEROS OF DIRICHLET L-FUNCTIONS

A. Some properties of Dirichlet L-functions

We now consider the generalization of the previous results to Dirichlet L-functions. Let us first introduce the basic ingredients and definitions regarding this class of functions, which are all well known [18]. Dirichlet L-series are defined as

$$L(z,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^z} \qquad (\Re(z) > 1) \tag{23}$$

where the arithmetic function $\chi(n)$ is a Dirichlet character. They can all be analytically continued to the entire complex plane, except for a simple pole at $z = 1$, and are then referred to as Dirichlet L-functions.

There are an infinite number of distinct Dirichlet characters which are primarily characterized by their modulus k, which determines their periodicity. They can be defined axiomatically, which leads to specific properties, some of which we now describe. Consider a Dirichlet character χ mod k, and let the symbol (n, k) denote the greatest common divisor of the two integers n and k. Then χ has the following properties:

1. $\chi(n + k) = \chi(n)$.

2. $\chi(1) = 1$ and $\chi(0) = 0$.

3. $\chi(nm) = \chi(n)\chi(m)$.

4. $\chi(n) = 0$ if $(n, k) > 1$ and $\chi(n) \neq 0$ if $(n, k) = 1$.

5. If $(n, k) = 1$ then $\chi(n)^{\varphi(k)} = 1$, where $\varphi(k)$ is the Euler totient arithmetic function. This implies that $\chi(n)$ are roots of unity.

6. If χ is a Dirichlet character so is the complex conjugate χ^*.

For a given modulus k there are $\varphi(k)$ distinct Dirichlet characters, which essentially follows from property 5 above. They can thus be labeled as $\chi_{k,j}$ where $j = 1, 2, \ldots, \varphi(k)$ denotes an arbitrary ordering. If $k = 1$ we have the *trivial* character where $\chi(n) = 1$ for every n, and (23) reduces to the Riemann ζ-function. The *principal* character, usually denoted by χ_1, is defined as $\chi_1(n) = 1$ if $(n, k) = 1$ and zero otherwise. In the above notation the principal character is always $\chi_{k,1}$.

Characters can be classified as *primitive* or *non-primitive*. Consider the Gauss sum

$$G(\chi) = \sum_{m=1}^{k} \chi(m) e^{2\pi i m/k}. \tag{24}$$

If the character χ mod k is primitive, then $|G(\chi)|^2 = k$. This is no longer valid for a non-primitive character. Consider a non-primitive character $\overline{\chi}$ mod $\overline{k}$. Then it can be expressed in terms of a primitive character of smaller modulus as $\overline{\chi}(n) = \overline{\chi_1}(n)\chi(n)$, where $\overline{\chi_1}$ is the principal character mod $\overline{k}$ and χ is a primitive character mod $k < \overline{k}$, where k is a divisor of $\overline{k}$. More precisely, k must be the *conductor* of $\overline{\chi}$ (see [18] for further details). In this case the two L-functions are related as $L(z, \overline{\chi}) = L(z, \chi)\Pi_{p|\overline{k}}\left(1 - \chi(p)/p^z\right)$. Thus $L(z, \overline{\chi})$ has the same zeros as $L(z,\chi)$. The principal character is only primitive when $k=1$, which yields the ζ-function. The simplest example of non-primitive characters are all the principal ones for $k\geq 2$, whose zeros are the same as the ζ-function. Let us consider another example with $\overline{k}=6$, where $\varphi(6)=2$, namely $\overline{\chi}_{6,2}$, whose components are [47]

$$\begin{array}{c|cccccc} n & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \overline{\chi}_{6,2}(n) & 1 & 0 & 0 & 0 & -1 & 0 \end{array} \tag{25}$$

In this case, the only divisors are 2 and 3. Since χ_1 mod 2 is non-primitive, it is excluded. We are left with $k=3$ which is the conductor of $\overline{\chi}_{6,2}$. Then we have two options; $\chi_{3,1}$ which is the non-primitive principal character mod 3, thus excluded, and $\chi_{3,2}$ which is primitive. Its components are

$$\begin{array}{c|ccc} n & 1 & 2 & 3 \\ \hline \chi_{3,2}(n) & 1 & -1 & 0 \end{array} \tag{26}$$

Note that $|G(\chi_{6,2})|^2=3\neq 6$ and $|G(\chi_{3,2})|^2=3$. In fact one can check that $\overline{\chi}_{6,2}(n)=\overline{\chi}_{6,1}(n)\chi_{3,2}(n)$, where $\overline{\chi}_{6,1}$ is the principal character mod $\overline{k}=6$. Thus the zeros of $L(z,\overline{\chi}_{6,2})$ are the same as those of $L(z,\chi_{3,2})$. Therefore, it suffices to consider primitive characters, and we will henceforth do so.

We will need the functional equation satisfied by $L(z,\chi)$. Let χ be a *primitive* character. Define its *order* a such that

$$a\equiv\begin{cases}1 & \text{if } \chi(-1)=-1 \text{ (odd)}\\ 0 & \text{if } \chi(-1)=1 \text{ (even)}\end{cases} \tag{27}$$

Let us define the meromorphic function

$$\Lambda(z,\chi)\equiv\left(\frac{k}{\pi}\right)^{\frac{z+a}{2}}\Gamma\left(\frac{z+a}{2}\right)L(z,\chi). \tag{28}$$

Then Λ satisfies the well known functional equation [18]

$$\Lambda(z,\chi)=\frac{i^{-a}\,G(\chi)}{\sqrt{k}}\,\Lambda(1-z,\chi^*). \tag{29}$$

The above equation is only valid for primitive characters.

B. Exact equation for the n-th zero

For a primitive character, since $|G(\chi)|=\sqrt{k}$, the factor on the right hand side of (29) is a phase. It is thus possible to obtain a more symmetric form through a new function defined as

$$\xi(z,\chi) \equiv \frac{i^{a/2}\, k^{1/4}}{\sqrt{G\,(\chi)}}\, \Lambda(z,\chi). \tag{30}$$

It then satisfies

$$\xi(z,\chi) = \xi^*(1-z,\chi) \equiv \left(\xi(1-z^*,\chi)\right)^* . \tag{31}$$

Above, the function ξ^* of z is defined as the complex conjugation of all coefficients that define ξ, namely χ and the $i^{a/2}$ factor, evaluated at a non-conjugated z.

Note that $(\Lambda(z,\chi))^* = \Lambda(z^*,\chi^*)$. Using the well known result $G\,(\chi^*) = \chi(-1)\,(G(\chi))^*$ we conclude that

$$\left(\xi(z,\chi)\right)^* = \xi\left(z^*,\chi^*\right). \tag{32}$$

This implies that if the character is real, then if ρ is a zero of ξ so is ρ^*, and one needs only consider ρ with positive imaginary part. On the other hand if $\chi \neq \chi^*$, then the zeros with negative imaginary part are different than ρ^*. For the trivial character where $k=1$ and $a=0$, implying $\chi(n)=1$ for any n, then $L(z,\chi)$ reduces to the Riemann ζ-function and (31) yields the well known functional equation (2).

Let $z = x+iy$. Then the function (30) can be written as $\xi(z,\chi) = Ae^{i\theta}$ where

$$A(x,y,\chi) = \left(\frac{k}{\pi}\right)^{\frac{x+a}{2}} \left|\Gamma\left(\frac{x+a+iy}{2}\right)\right| \left|L(x+iy,\chi)\right|, \tag{33}$$

$$\theta(x,y,\chi) = \arg\Gamma\left(\frac{x+a+iy}{2}\right) - \frac{y}{2}\log\left(\frac{\pi}{k}\right) - \frac{1}{2}\arg G(\chi) + \arg L(x+iy,\chi) + \frac{\pi a}{4}. \tag{34}$$

From (32) we also conclude that $A(x,y,\chi) = A(x,-y,\chi^*)$ and $\theta(x,y,\chi) = -\theta(x,-y,\chi^*)$. Denoting $\xi^*(1-z,\chi) = A'e^{-i\theta'}$ we have $A'(x,y,\chi) = A(1-x,y,\chi)$ and $\theta'(x,y,\chi) = \theta(1-x,y,\chi)$. Taking the modulus of (31) we also have that $A(x,y,\chi) = A'(x,y,\chi)$ for any z.

On the critical strip, the functions $L(z,\chi)$ and $\xi(z,\chi)$ have the same zeros. Thus on a zero we clearly have

$$\lim_{\delta\to 0^+} \left\{\xi(\rho+\delta,\chi) + \xi^*(1-\rho-\delta,\chi)\right\} = 0. \tag{35}$$

Let us define

$$B(x,y,\chi) \equiv e^{i\theta(x,y,\chi)} + e^{-i\theta'(x,y,\chi)}. \tag{36}$$

Since $A = A'$ everywhere, from (35) we conclude that on a zero we have

$$\lim_{\delta\to 0^+} A(x+\delta,y,\chi)B(x+\delta,y,\chi) = 0. \tag{37}$$

As before, let us consider the particular solution of $\lim_{\delta\to 0^+} B = 0$ given by

$$\theta = \theta', \qquad \lim_{\delta\to 0^+} \cos\theta = 0. \tag{38}$$

Let us define the function

$$\begin{aligned} \vartheta_{k,a}(y) &\equiv \arg\Gamma\left(\frac{1}{4} + \frac{a}{2} + i\,\frac{y}{2}\right) - \frac{y}{2}\log\left(\frac{\pi}{k}\right) \\ &= \Im\left[\log\Gamma\left(\frac{1}{4} + \frac{a}{2} + i\,\frac{y}{2}\right)\right] - \frac{y}{2}\log\left(\frac{\pi}{k}\right). \end{aligned} \tag{39}$$

When $k = 1$ and $a = 0$, the function (39) is just the usual Riemann-Siegel ϑ function (18). Since the function $\log\Gamma$ has a complicated branch cut, one can use the following series representation in (39) [40]

$$\log\Gamma(z) = -\gamma z - \log z - \sum_{n=1}^{\infty}\left\{\log\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right\}, \tag{40}$$

where γ is the Euler-Mascheroni constant. Nevertheless, most numerical packages already have the $\log\Gamma$ function implemented.

On the critical line the first equation in (38) is already satisfied. From the second equation we have $\lim_{\delta\to 0^+}\theta\left(\tfrac{1}{2} + \delta, y\right) = \left(n + \tfrac{1}{2}\right)\pi$, therefore

$$\vartheta_{k,a}(y_n) + \lim_{\delta\to 0^+}\arg L\left(\tfrac{1}{2} + \delta + iy_n, \chi\right) - \frac{1}{2}\arg G(\chi) + \frac{\pi a}{4} = \left(n + \frac{1}{2}\right)\pi. \tag{41}$$

Analyzing the left hand side of (41) we can see that it has a minimum, thus we shift $n \to n - (n_0 + 1)$ for a given n_0, to label the zeros according to the convention that the first positive zero is labelled by $n = 1$. Thus the upper half of the critical line will have the zeros labelled by $n = 1, 2, \ldots$ corresponding to positive y_n, while the lower half will have the negative values y_n labelled by $n = 0, -1, \ldots$. The integer n_0 depends on k, a and χ, and should be chosen according to each specific case. In the cases we analyze below $n_0 = 0$, whereas for the trivial character $n_0 = 1$. In practice, the value of n_0 can always be determined by plotting (41) with $n = 1$, passing all terms to its left hand side. Then it is trivial to adjust the integer n_0 such that the graph passes through the point $(y_1, 0)$ for the first jump, corresponding to the first positive solution. Henceforth we will *omit* the integer n_0 in the equations, since all cases analyzed in the following have $n_0 = 0$. Nevertheless, the reader should bear in mind that for other cases, it may be necessary to replace $n \to n - n_0$ in the following equations.

In summary, these zeros have the form $\rho_n = \frac{1}{2} + iy_n$, where for a given $n \in \mathbb{Z}$, the imaginary part y_n is the solution of the equation

$$\vartheta_{k,a}(y_n) + \lim_{\delta \to 0^+} \arg L\left(\tfrac{1}{2} + \delta + iy_n, \chi\right) - \frac{1}{2} \arg G(\chi) = \left(n - \frac{1}{2} - \frac{a}{4}\right)\pi. \tag{42}$$

C. Asymptotic equation for the n-th zero

From Stirling's formula we have the following asymptotic form for $y \to \pm\infty$:

$$\vartheta_{k,a}(y) = \operatorname{sgn}(y)\left(\frac{|y|}{2}\log\left(\frac{k|y|}{2\pi e}\right) + \frac{2a-1}{8}\pi + O(1/y)\right). \tag{43}$$

The first order approximation of (42), i.e. neglecting terms of $O(1/y)$, is therefore given by

$$\sigma_n \frac{|y_n|}{2\pi}\log\left(\frac{k\,|y_n|}{2\pi e}\right) + \frac{1}{\pi}\lim_{\delta\to 0^+}\arg L\left(\tfrac{1}{2} + \delta + i\sigma_n|y_n|, \chi\right) - \frac{1}{2\pi}\arg G(\chi) = n + \frac{\sigma_n - 4 - 2a(1+\sigma_n)}{8}, \tag{44}$$

where $\sigma_n = 1$ if $n > 0$ and $\sigma_n = -1$ if $n \le 0$. For $n > 0$ we have $y_n = |y_n|$ and for $n \le 0$ $y_n = -|y_n|$.

D. Counting formulas

Let us define $N_0^+(T,\chi)$ as the number of zeros on the critical line with $0 < \Im(\rho) < T$ and $N_0^-(T,\chi)$ as the number of zeros with $-T < \Im(\rho) < 0$. As explained before, $N_0^+(T,\chi) \neq N_0^-(T,\chi)$ if the characters are complex numbers, since the zeros are not symmetrically distributed between the upper and lower half of the critical line.

The counting formula $N_0^+(T,\chi)$ is obtained from (42) by replacing $y_n \to T$ and $n \to N_0^+ + 1/2$, therefore

$$N_0^+(T,\chi) = \frac{1}{\pi}\vartheta_{k,a}(T) + \frac{1}{\pi}\arg L\left(\tfrac{1}{2} + iT, \chi\right) - \frac{1}{2\pi}\arg G(\chi) + \frac{a}{4}. \tag{45}$$

The passage from (42) to (45) is justified under the assumptions already discussed in connection with (13) and (20), i.e. assuming that (42) has a unique solution for every n. Analogously, the counting formula on the lower half line is given by

$$N_0^-(T,\chi) = \frac{1}{\pi}\vartheta_{k,a}(T) - \frac{1}{\pi}\arg L\left(\tfrac{1}{2} - iT, \chi\right) + \frac{1}{2\pi}\arg G(\chi) - \frac{a}{4}. \tag{46}$$

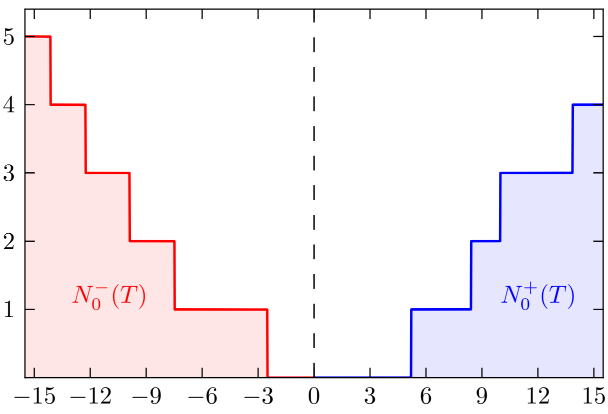

FIG. 2: Exact counting formulae (45) and (46). Note that they are not symmetric with respect to the origin, since the L-zeros for complex χ are not complex conjugates. We used $\chi = \chi_{7,2}$ (76).

Note that in (45) and (46) T is positive. Both cases are plotted in FIG. 2 for the character $\chi_{7,2}$ shown in (76). One can notice that they are precisely staircase functions, jumping by one at each zero. Note also that the functions are not symmetric about the origin, since for a complex χ the zeros on upper and lower half lines are not simply complex conjugates.

From (43) we also have the first order approximation for $T \to \infty$,

$$N_0^+(T,\chi) = \frac{T}{2\pi}\log\left(\frac{k\,T}{2\pi e}\right) + \frac{1}{\pi}\arg L\left(\tfrac{1}{2}+iT,\chi\right) - \frac{1}{2\pi}\arg G\left(\chi\right) - \frac{1}{8} + \frac{a}{2}. \tag{47}$$

Analogously, for the lower half line we have

$$N_0^-(T,\chi) = \frac{T}{2\pi}\log\left(\frac{k\,T}{2\pi e}\right) - \frac{1}{\pi}\arg L\left(\tfrac{1}{2}-iT,\chi\right) + \frac{1}{2\pi}\arg G\left(\chi\right) - \frac{1}{8}. \tag{48}$$

As in (42), again we are omitting n_0 since in the cases below $n_0 = 0$, but for other cases one may need to include $\pm n_0$ on the right hand side of $N_0^\pm$, respectively.

It is known that the number of zeros on the *entire critical strip* up to height T, i.e. in the region $\{0 < x < 1,\, 0 < y < T\}$, is given by [8]

$$N^+(T,\chi) = \frac{1}{\pi}\vartheta_{k,a}\left(T\right) + \frac{1}{\pi}\arg L\left(\tfrac{1}{2}+iT,\chi\right) - \frac{1}{\pi}\arg L\left(\tfrac{1}{2},\chi\right). \tag{49}$$

From Stirling's approximation and noticing that $2a-1 = -\chi(-1)$, for $T \to \infty$ we obtain the asymptotic approximation [8, 19]

$$N^+(T,\chi) = \frac{T}{2\pi}\log\left(\frac{k\,T}{2\pi e}\right) + \frac{1}{\pi}\arg L\left(\tfrac{1}{2}+iT,\chi\right) - \frac{1}{\pi}\arg L\left(\tfrac{1}{2},\chi\right) - \frac{\chi(-1)}{8} + O(1/T). \tag{50}$$

Both formulas (49) and (50) are exactly the same as (45) and (47), respectively. This can be seen as follows. From (31) we conclude that ξ is real on the critical line. Thus $\arg \xi\left(\frac{1}{2}\right) = 0 = -\frac{1}{2}\arg G(\chi) + \arg L\left(\frac{1}{2},\chi\right) + \frac{\pi a}{4}$. Then, replacing $\arg G$ in (42) we obtain

$$\vartheta_{k,a}(y_n) + \lim_{\delta\to 0^+} \arg L\left(\tfrac{1}{2}+\delta+iy_n,\chi\right) - \arg L\left(\tfrac{1}{2},\chi\right) = \left(n-\tfrac{1}{2}\right)\pi. \tag{51}$$

Replacing $y_n \to T$ and $n \to N_0^+ + 1/2$ in (51) we have precisely the expression (49), and also (50) for $T\to\infty$. Therefore, we conclude that $N_0^+(T,\chi) = N^+(T,\chi)$ exactly. From (32) we see that negative zeros for character χ correspond to positive zeros for character χ^*. Then for $-T < \Im(\rho) < 0$ the counting on the strip also coincides with the counting on the line, since $N_0^-(T,\chi) = N_0^+(T,\chi^*)$ and $N^-(T,\chi) = N^+(T,\chi^*)$. Therefore, the number of zeros on the whole *critical strip* is the same as the number of zeros on the *critical line* obtained as solutions of (42). This is valid under the assumption that (42) has a unique solution for every n.

IV. ZEROS OF *L*-FUNCTIONS BASED ON MODULAR FORMS

Let us generalize the previous results to L-functions based on level one modular forms. We first recall some basic definitions and properties. The *modular group* can be represented by the set of 2×2 integer matrices,

$$SL_2(\mathbb{Z}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d\in\mathbb{Z},\ \det A = 1 \right\}, \tag{52}$$

provided each matrix A is identified with $-A$, i.e. $\pm A$ are regarded as the same transformation. Thus for τ in the upper half complex plane, it transforms as $\tau \mapsto A\tau = \frac{a\tau+b}{c\tau+d}$ under the action of the modular group. A *modular form* f of weight k is a function that is analytic in the upper half complex plane which satisfies the functional relation [43]

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau). \tag{53}$$

If the above equation is satisfied for all of $SL_2(\mathbb{Z})$, then f is referred to as being of level one. It is possible to define higher level modular forms which satisfy the above equation for a subgroup of $SL_2(\mathbb{Z})$. Since our results are easily generalized to the higher level case, henceforth we will only consider level one forms.

For the $SL_2(\mathbb{Z})$ element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the above implies the periodicity $f(\tau) = f(\tau+1)$, thus it has a Fourier series

$$f(\tau) = \sum_{n=0}^{\infty} a_f(n)\, q^n, \qquad q \equiv e^{2\pi i \tau}. \tag{54}$$

If $a_f(0) = 0$ then f is called a *cusp form*.

From the Fourier coefficients, one can define the Dirichlet series

$$L_f(z) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^z}. \tag{55}$$

The functional equation for $L_f(z)$ relates it to $L_f(k-z)$, so that the critical line is $\Re(z) = \frac{k}{2}$, where $k \geq 4$ is an even integer. One can always shift the critical line to $\frac{1}{2}$ by replacing $a_f(n)$ by $a_f(n)/n^{(k-1)/2}$, however we will not do this here. Let us define

$$\Lambda_f(z) \equiv (2\pi)^{-z}\, \Gamma(z)\, L_f(z). \tag{56}$$

Then the functional equation is given by [43]

$$\Lambda_f(z) = (-1)^{k/2} \Lambda_f(k-z). \tag{57}$$

There are only two cases to consider since $\frac{k}{2}$ can be an even or an odd integer. As in (30) we can absorb the extra minus sign factor for the odd case. Thus we define $\xi_f(z) \equiv \Lambda_f(z)$ for $\frac{k}{2}$ even, and we have $\xi_f(z) = \xi_f(k-z)$, and $\xi_f(z) \equiv e^{-i\pi/2}\Lambda_f(z)$ for $\frac{k}{2}$ odd, implying $\xi_f(z) = \xi_f^*(k-z)$. Representing $\xi_f(z) = |\xi_f|\, e^{i\theta}$ where $z = x+iy$, we follow exactly the same steps as in the previous sections. From the particular solution (38) we conclude that there are infinite zeros on the critical line $\Re(\rho) = \frac{k}{2}$ determined by $\lim_{\delta\to 0^+} \theta\left(\frac{k}{2}+\delta, y, \chi\right) = \left(n - \frac{1}{2}\right)\pi$. Therefore, these zeros have the form $\rho_n = \frac{k}{2} + i y_n$, where y_n is the solution of the equation

$$\vartheta_k(y_n) + \lim_{\delta\to 0^+} \arg L_f\left(\tfrac{k}{2}+\delta+iy_n\right) = \left(n - \frac{1+(-1)^{k/2}}{4}\right)\pi \qquad (n = 1, 2, \dots) \tag{58}$$

where we have defined

$$\vartheta_k(y) \equiv \arg\Gamma\left(\tfrac{k}{2}+iy\right) - y\log 2\pi. \tag{59}$$

This implies that the number of solutions of (58) with $0 < y < T$ is given by

$$N_0(T) = \frac{1}{\pi}\vartheta_k(T) + \frac{1}{\pi}\arg L_f\left(\tfrac{k}{2}+iT\right) - \frac{1-(-1)^{k/2}}{4}. \tag{60}$$

In the limit of large y_n, neglecting terms of $O(1/y)$, the equation (58) becomes

$$y_n \log\left(\frac{y_n}{2\pi e}\right) + \lim_{\delta\to 0^+} \arg L_f\left(\tfrac{k}{2}+\delta+iy_n\right) = \left(n - \frac{k+(-1)^{k/2}}{4}\right)\pi \qquad (n = 1, 2, \dots). \tag{61}$$

V. APPROXIMATE ZEROS IN TERMS OF THE LAMBERT *W*-FUNCTION

A. Explicit formula

We now show that it is possible to obtain an approximate solution to the previous transcendental equations with an explicit formula. Let us introduce the Lambert W-function [35], which is defined for any complex number z through the equation

$$W(z)e^{W(z)} = z. \tag{62}$$

The multi-valued W-function cannot be expressed in terms of other known elementary functions. If we restrict attention to real-valued $W(x)$ there are two branches. The principal branch occurs when $W(x) \geq -1$ and is denoted by W_0, or simply W for short, and its domain is $x \geq -1/e$. The secondary branch, denoted by W_{-1}, satisfies $W_{-1}(x) \leq -1$ for $-e^{-1} \leq x < 0$. Since we are interested only in positive real-valued solutions, we just need the principal branch where W is single-valued.

Let us start with the zeros of the ζ-function, described by equation (12). Consider its leading order approximation, or equivalently its average since $\langle \arg \zeta\left(\frac{1}{2}+iy\right)\rangle = 0$. Then we have the transcendental equation

$$\frac{\widetilde{y}_n}{2\pi}\log\left(\frac{\widetilde{y}_n}{2\pi e}\right) = n - \frac{11}{8}. \tag{63}$$

Through the transformation $\widetilde{y}_n = 2\pi\left(n-\frac{11}{8}\right)x_n^{-1}$, this equation can be written as $x_n e^{x_n} = e^{-1}\left(n-\frac{11}{8}\right)$. Comparing with (62) we thus we obtain

$$\widetilde{y}_n = \frac{2\pi\left(n-\frac{11}{8}\right)}{W\left[e^{-1}\left(n-\frac{11}{8}\right)\right]} \qquad (n = 1, 2, \dots). \tag{64}$$

Although the inversion from (63) to (64) is rather simple, it is very convenient since it is indeed an explicit formula depending only on n, and W is included in most numerical packages. It gives an approximate solution for the ordinates of the Riemann zeros in closed form. The values computed from (64) are much closer to the Riemann zeros than Gram points, and one does not have to deal with violations of Gram's law (see below).

Analogously, for Dirichlet L-functions, after neglecting the $\arg L$ term, the equation (44) yields a transcendental equation which can be writen as $x_n e^{x_n} = kA_n e^{-1}$ through the transformation $|y_n| = 2\pi A_n x_n^{-1}$, where

$$A_n(\chi) = \sigma_n\left(n + \frac{1}{2\pi}\arg G(\chi)\right) + \frac{1-4\sigma_n-2a\left(\sigma_n+1\right)}{8}. \tag{65}$$

Thus the approximate solution is explicitly given by

$$\widetilde{y}_n = \frac{2\pi\sigma_n A_n\left(\chi\right)}{W\left[k\, e^{-1} A_n\left(\chi\right)\right]} \qquad (n = 0, \pm 1, \pm 2, \dots). \tag{66}$$

In the above formula $n = 1, 2, \dots$ correspond to positive y_n solutions, while $n = 0, -1, \dots$ correspond to negative y_n solutions. Contrary to the ζ-function, in general, the zeros are not conjugate related along the critical line.

In the same way, ignoring the small $\arg L_f$ term in (61), the approximate solution for the imaginary part of the zeros of L-functions based on level one modular forms is given by

$$\widetilde{y}_n = \frac{A_n \pi}{W\left[(2e)^{-1} A_n\right]} \qquad (n = 1, 2, \dots). \tag{67}$$

where

$$A_n = n - \frac{k + (-1)^{k/2}}{4}. \tag{68}$$

B. Further remarks

Let us focus on the approximation (64) regarding zeros of the ζ-function. Obviously the same arguments apply to the zeros of the other classes of functions, based on formulas (66) and (67).

Remark 6. The estimates given by (64) can be calculated to high accuracy for arbitrarily large n, since W is a standard elementary function. Of course, the $\widetilde{y}_n$ are not as accurate as the solutions y_n including the $\arg \zeta$ term, as we will see in section VI. Nevertheless, it is indeed a good estimate, especially if one considers very high zeros, where traditional methods have not previously estimated such high values. For instance, formula (64) can easily estimate the zeros shown in TABLE I, and much higher if desirable. The numbers in this table are accurate approximations to the n-th zero to the number of digits shown, which is approximately the number of digits in the integer part. For instance, the approximation to the 10^{100} zero is correct to 100 digits. With Mathematica we easily calculated the first million digits of the 10^{10^6} zero.

Remark 7. Using the asymptotic behaviour $W(x) \sim \log x$ for large x, the n-th zero is approximately $\widetilde{y}_n \approx 2\pi n / \log n$, as already known [5]. The distance between consecutive zeros is $2\pi / \log n$, which tends to zero when $n \to \infty$.

n	$\widetilde{y}_n$
$10^{22}+1$	$1.370919909931995308226770 \times 10^{21}$
10^{50}	$5.741532903784313725642221053588442131126693322343461 \times 10^{48}$
10^{100}	$2.80690383842894069903195445838256400084548030162846045192360059224930$ $92234907304306033565310925247323 4 \times 10^{98}$
10^{200}	$1.38579222214678934084546680546715919012340245153870708183286835248393$ $89096897963430767976394081726100286517919948794007280268632988409580 91$ $288304951600695814960962282888090054696215023267048447330585768 \times 10^{198}$

TABLE I: Formula (64) can easily estimate very high Riemann zeros. The results are expected to be correct up to the decimal point, i.e. to the number of digits in the integer part. The numbers are shown with three digits beyond the integer part.

Remark 8. The solutions (64) are reminiscent of the so-called Gram points g_n, which are solutions to $\vartheta(g_n) = n\pi$ where ϑ is given by (18). Gram's law is the tendency for Riemann zeros to lie between consecutive Gram points, but it is known to fail for about 1/4 of all Gram intervals. Our $\widetilde{y}_n$ are intrinsically different from Gram points. It is an approximate solution for the ordinate of the zero itself. In particular, the Gram point $g_0 = 17.8455$ is the closest to the first Riemann zero, whereas $\widetilde{y}_1 = 14.52$ is already much closer to the true zero which is $y_1 \approx 14.1347$. The traditional method to compute the zeros is based on the Riemann-Siegel formula, $\zeta\left(\frac{1}{2}+it\right) = Z(t)\left(\cos\vartheta(t) - i\sin\vartheta(t)\right)$, and the empirical observation that the real part of this equation is almost always positive, except when Gram's law fails, and $Z(t)$ has the opposite sign of $\sin\vartheta$. Since $Z(t)$ and $\zeta\left(\frac{1}{2}+it\right)$ have the same zeros, one looks for the zeros of $Z(t)$ between two Gram points, as long as Gram's law holds $(-1)^n Z\left(g_n\right) > 0$. To verify the RH numerically, the counting formula (21) must also be used, to assure that the number of zeros on the critical line coincide with the number of zeros on the strip. The detailed procedure is throughly explained in [1, 5]. Based on this method, amazingly accurate solutions and high zeros on the critical line were computed [29, 33, 37, 39]. Nevertheless, our proposal is *fundamentally* different. We claim that (19), or its asymptotic approximation (12), is the equation that determines the Riemann zeros on the critical line. Then, one just needs to find its solution for a given n. We will compute the Riemann zeros in this way in the next section, just by solving the equation numerically, starting from the approximation given by the explicit formula (64), without using Gram points nor the Riemann-Siegel Z function. Let us emphasize that our goal is not to provide

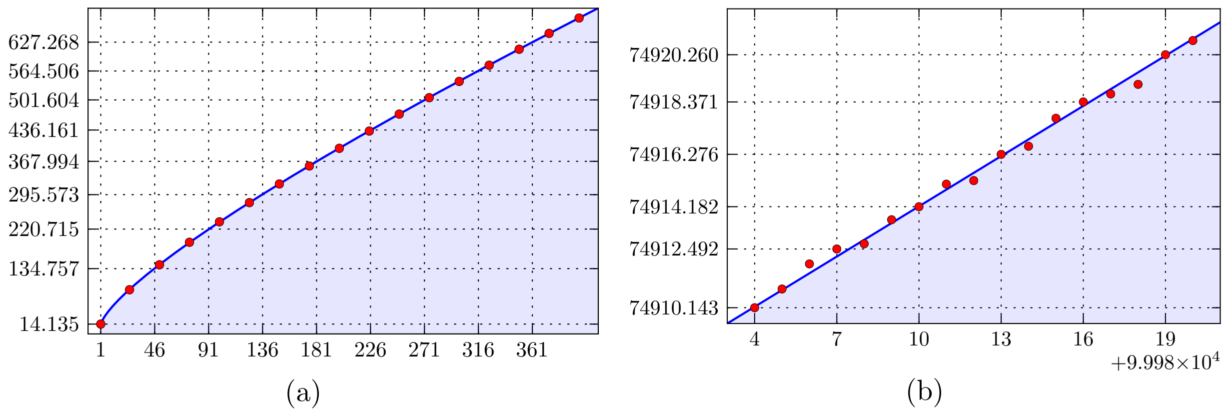

FIG. 3: Comparison of the prediction of (64) (blue line) and (12) (red dots). We are ploting y_n against n. (a) $n \in [1, \dots, 400]$. Note how the solutions are close at first sight. (b) If we focus on a small range we can see how the solutions of (12) oscillate around the line (64) due to the fluctuating term $\arg\zeta$. Here $n \in [99984, \dots, 10^5]$.

a more efficient algorithm to compute the zeros [37], although the method described here may very well be, but to justify the validity of equations (12) and (19).

VI. NUMERICAL ANALYSIS: ζ-FUNCTION

A. Importance of the $\arg\zeta$ term.

Instead of solving the exact equation (19) we will initially consider its first order approximation, which is equation (12). As we will see, this approximation already yields surprisingly accurate values for the Riemann zeros.

Let us first consider how the approximate solution given by (64) is modified by the presence of the $\arg\zeta$ term in (12). Numerically, we compute $\arg\zeta$ taking its principal value. As already discussed in Remark 1, the function $\arg\zeta\left(\frac{1}{2}+iy\right)$ oscillates around zero and changes sign in the vicinity of each Riemann zero, as shown in FIG. 1a. At a zero it can be well-defined by the limit (6), which is generally not zero. For example, for the first Riemann zero $y_1 \approx 14.1347$, $\lim_{\delta\to 0^+} \arg\zeta\left(\frac{1}{2}+\delta+iy_1\right) = 0.1578739$. The $\arg\zeta$ term plays an important role and indeed improves the estimate of the n-th zero. This can be seen in FIG. 3, where we compare the estimate given by (64) with the numerical solutions of (12).

Since equation (12) alternates in sign around a zero, we can apply a root finder method in an appropriate interval, centered around the approximate solution $\widetilde{y}_n$ given by formula

n	$\widetilde{y}_n$	y_n
1	14.52	14.134725142
10	50.23	49.773832478
10^2	235.99	236.524229666
10^3	1419.52	1419.422480946
10^4	9877.63	9877.782654006
10^5	74920.89	74920.827498994
10^6	600269.64	600269.677012445
10^7	4992381.11	4992381.014003179
10^8	42653549.77	42653549.760951554
10^9	371870204.05	371870203.837028053
10^{10}	3293531632.26	3293531632.397136704

TABLE II: Numerical solutions of equation (12). All numbers shown are accurate up to the 9-th decimal place and agree with [33, 36].

n	y_n
1	14.134725141734693790457251983562470
2	21.022039638771554992628479593896902
3	25.010857580145688763213790992562821
4	30.424876125859513210311897530584091
5	32.935061587739189690662368964074903

TABLE III: Numerical solutions to (12) for the lowest zeros. Although it was derived for high y, it provides accurate numbers even for the lower zeros. This numbers are correct up to the decimal place shown [33].

(64). Some of the solutions obtained in this way are presented in TABLE II, and are accurate up to the number of decimal places shown. We used only Mathematica or some very simple algorithms to perform these numerical computations, taken from standard open source numerical libraries.

Although the formula for y_n was derived for large n, it is surprisingly accurate even for the lower zeros, as shown in TABLE III. It is actually easier to solve numerically for low zeros since $\arg \zeta$ is better behaved. These numbers are correct up to the number of digits shown, and the precision was improved simply by decreasing the error tolerance.

B. GUE Statistics

The link between the Riemann zeros and random matrix theory started with the pair correlation of zeros, proposed by Montgomery [8], and the observation of F. Dyson that it is the same as the 2-point correlation function predicted by the gaussian unitary ensemble (GUE) for large random matrices [38].

The main purpose of this section is to test whether our approximation (12) to the zeros is accurate enough to reveal this statistics. Whereas formula (64) is a valid estimate of the zeros, it is not sufficiently accurate to reproduce the GUE statistics, since it does not have the oscillatory $\arg\zeta$ term. On the other hand, the solutions to equation (12) are accurate enough, which again indicates the importance of the $\arg\zeta$.

Montgomery's pair correlation conjecture can be stated as follows:

$$\frac{1}{N(T)} \sum_{\substack{0\le y,y'\le T \\ \alpha<\frac{1}{2\pi}\log\left(\frac{T}{2\pi}\right)(y-y')\le\beta}} 1 \quad \sim \quad \int_{\alpha}^{\beta} du \left(1-\frac{\sin^2(\pi u)}{\pi^2u^2}\right) \tag{69}$$

where $0<\alpha<\beta$, $N(T)\sim\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right)$ according to (14), and the statement is valid in the limit $T\to\infty$. The right hand side of (69) is the 2-point GUE correlation function. The average spacing between consecutive zeros is given by $\frac{T}{N}\sim 2\pi/\log\left(\frac{T}{2\pi}\right)\to 0$ as $T\to\infty$. This can also be seen from (64) for very large n, i.e. $\widetilde{y}_{n+1}-\widetilde{y}_n\to 0$ as $n\to\infty$. Thus the distance between zeros on the left hand side of (69), under the sum, is a normalized distance.

While (69) can be applied if we start from the first zero on the critical line, it is unable to provide a test if we are centered around a given high zero on the line. To deal with such a situation, Odlyzko [39] proposed a stronger version of Montgomery's conjecture, by taking into account the large density of zeros higher on the line. This is done by replacing the normalized distance in (69) by a sum of normalized distances over consecutive zeros in the form

$$\delta_n \equiv \frac{1}{2\pi}\log\left(\frac{y_n}{2\pi}\right)(y_{n+1}-y_n)\,. \tag{70}$$

Thus (69) is replaced by

$$\frac{1}{(N-M)(\beta-\alpha)} \sum_{\substack{M\le m,n\le N \\ \alpha<\sum_{k=1}^{n}\delta_{m+k}\le\beta}} 1 \quad = \quad \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} du\left(1-\frac{\sin^2(\pi u)}{\pi^2u^2}\right), \tag{71}$$

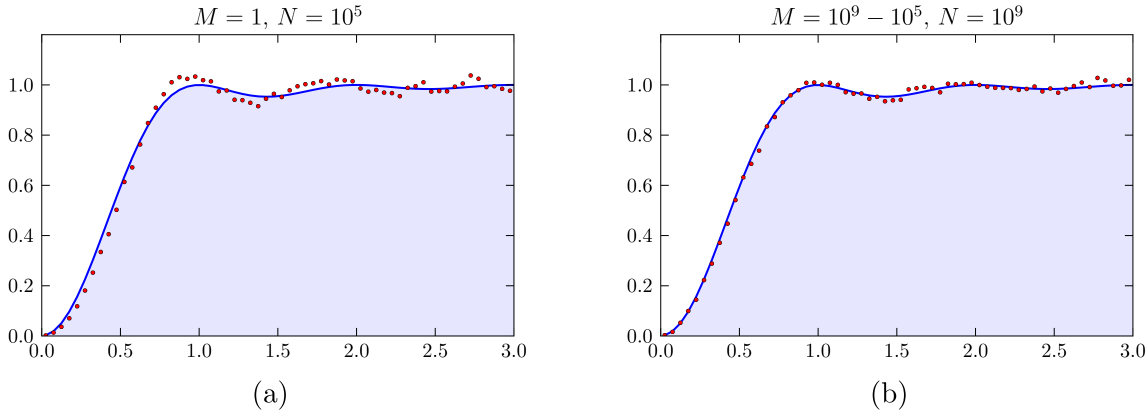

FIG. 4: The solid line represents the right hand side of (71) and the dots represent its left hand side, computed from equation (12). The parameters are $\beta = \alpha + 0.05$, $\alpha = (0, 0.05, \ldots, 3)$ and the x-axis is given by $x = \frac{1}{2}(\alpha + \beta)$. (a) We use the first 10^5 zeros. (b) The same parameters but using zeros in the middle of the critical line; $M = 10^9 - 10^5$ and $N = 10^9$.

where M is the label of a given zero on the line and $N > M$. In this sum it is assumed that $n > m$ also, and we included the correct normalization on both sides. The conjecture (71) is already well supported by extensive numerical analysis [29, 39].

Odlyzko's conjecture (71) is a very strong constraint on the statistics of the zeros. Thus we submit the numerical solutions of equation (12), as discussed in the previous section, to this test. In FIG. 4a we can see the result for $M = 1$ and $N = 10^5$, with α ranging from $0 \ldots 3$ in steps of $s = 0.05$, and $\beta = \alpha + s$ for each value of α, i.e. $\alpha = (0.00, 0.05, 0.10, \ldots, 3.00)$ and $\beta = (0.05, 0.10, \ldots, 3.05)$. We compute the left hand side of (71) for each pair (α, β) and plot the result against $x = \frac{1}{2}(\alpha + \beta)$. In FIG. 4b we do the same thing but with $M = 10^9 - 10^5$ and $N = 10^9$. Clearly, the numerical solutions of (12) reproduce the correct statistics. In fact, FIG. 4a is identical to the one in [39]. The last zeros in these ranges are shown in TABLE IV.

C. Prime number counting function

In this section we explore whether our approximations to the Riemann zeros are accurate enough to reconstruct the prime number counting function. As usual, let $\pi(x)$ denote the number of primes less than x. Riemann obtained an explicit expression for $\pi(x)$ in terms of the non-trivial zeros of $\zeta(z)$. There are simpler but equivalent versions of the main result,

n	y_n
10^5-5	74917.719415828
10^5-4	74918.370580227
10^5-3	74918.691433454
10^5-2	74919.075161121
10^5-1	74920.259793259
10^5	74920.827498994

n	y_n
10^9-5	371870202.244870467
10^9-4	371870202.673284457
10^9-3	371870203.177729799
10^9-2	371870203.274345928
10^9-1	371870203.802552324
10^9	371870203.837028053

TABLE IV: Last numerical solutions to (12) around $n = 10^5$ and $n = 10^9$.

based on the function $\psi(x)$ below. However, let us present the main formula for $\pi(x)$ itself, since it is historically more important.

The function $\pi(x)$ is related to another number-theoretic function $J(x)$, defined as

$$J(x) = \sum_{2\le n\le x} \frac{\Lambda(n)}{\log n} \tag{72}$$

where $\Lambda(n)$, the von Mangoldt function, is equal to $\log p$ if $n = p^m$ for some prime p and an integer m, and zero otherwise. The two functions $\pi(x)$ and $J(x)$ are related by Möbius inversion:

$$\pi(x) = \sum_{n\ge 1} \frac{\mu(n)}{n} J(x^{1/n}). \tag{73}$$

Here, $\mu(n)$ is the Möbius function, equal to 1 (-1) if n is a product of an even (odd) number of distinct primes, and equal to zero if it has a multiple prime factor. The above expression is actually a finite sum, since for large enough n, $x^{1/n} < 2$ and $J = 0$.

The main result of Riemann is a formula for $J(x)$, expressed as an infinite sum over zeros ρ of the $\zeta(z)$ function:

$$J(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}\left(x^{\rho}\right) + \int_x^{\infty} \frac{dt}{\log t}\, \frac{1}{t\,(t^2-1)} - \log 2, \tag{74}$$

where $\mathrm{Li}(x) = \int_0^x dt/\log t$ is the log-integral function [48]. The above sum is real because the ρ's come in conjugate pairs. If there are no zeros on the line $\Re(z) = 1$, then the dominant term is the first one in the above equation, $J(x) \sim \mathrm{Li}(x)$, and this was used to prove the prime number theorem by Hadamard and de la Vallée Poussin.

The function $\psi(x)$ has the simpler form

$$\psi(x) = \sum_{n\le x} \Lambda(n) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right). \tag{75}$$

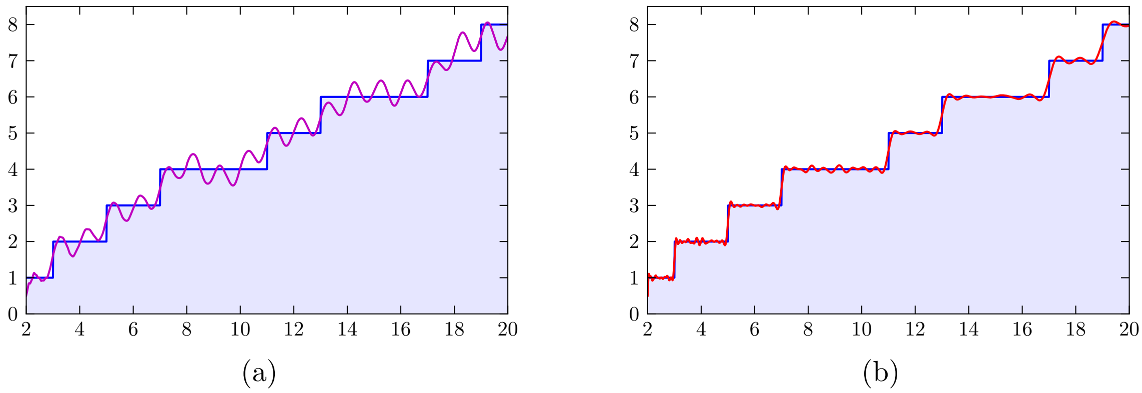

FIG. 5: The prime number counting function $\pi(x)$ with the first 50 Riemann zeros. (a) Zeros approximated by the formula (64). (b) Zeros obtained from numerical solutions to the equation (12).

In this formulation, the prime number theorem follows from the fact that the leading term is $\psi(x) \sim x$.

In Figure FIG. 5a we plot $\pi(x)$ from equations (73) and (74), computed with the first 50 zeros in the approximation $\rho_n = \frac{1}{2} + i\widetilde{y}_n$ given by (64). FIG. 5b shows the same plot with zeros obtained from the numerical solutions of equation (12). Although with the approximation $\widetilde{y}_n$ the curve is trying to follow the steps in $\pi(x)$, once again, one clearly sees the importance of the $\arg\zeta$ term.

D. Solutions to the exact equation

In the previous sections we have computed numerical solutions of (12) showing that, actually, this first order approximation to (19) is very good and already captures some interesting properties of the Riemann zeros, like the GUE statistics and ability to reproduce the prime number counting formula. Nevertheless, by simply solving (19) it is possible to obtain values for the zeros as accurately as desirable. The numerical procedure is performed as follows:

1. We apply a root finder method on (19) looking for the solution in a region centered around the number $\widetilde{y}_n$ provided by (64), with a not so small δ, for instance $\delta \sim 10^{-5}$.

2. We solve (19) again but now centered around the solution obtained in step 1 above, and we decrease δ, for instance $\delta \sim 10^{-8}$.

n	y_n
1	14.134725141734693790457251983562470270784257115699243175685 5
2	21.022039638771554992628479593896902777334340524902781754629 5
3	25.010857580145688763213790992562821818659549672557996672496 5
4	30.424876125859513210311897530584091320181560023715440180962 1
5	32.935061587739189690662368964074903488812715603517039009280 0

TABLE V: The first few numerical solutions to (19), accurate to 60 digits (58 decimals). These solutions were obtained using the root finder function in Mathematica (see Appendix B)

.

3. We repeat the procedure in step 2 above, decreasing δ again.

4. Through successive iterations, and decreasing δ each time, it is possible to obtain solutions as accurate as desirable. In carrying this out, it is important to not allow δ to be exactly zero.

An actual implementation of the above procedure in Mathematica is shown in Appendix B.

The first few zeros computed in this way are shown in TABLE V. Through successive iterations it is possible achieve even much higher accuracy than shown in TABLE V.

It is known that the first zero where Gram's law fails is for $n = 126$. Applying the same method, like for any other n, the solution of (19) starting with the approximation (64) does not present any difficulty. We easily found the following number:

$$y_{126} = 279.2292509277451892284098804519553592834926374055612935947 27$$

Just to illustrate, and to convince the reader, how the solutions of (19) can be made arbitrarily precise, we compute the zero $n = 1000$ accurate up to 500 decimal places, also using the same simple approach [49]:

$$\begin{aligned} y_{1000} = 1419.&422480945995686465989038079916819232100601064166016304690814684 60\\ &867641759301041791134329117920998748098423226056011874139744795 26\\ &506370672508342889831518454476882525931159442394251954846877081 63\\ &946256332381457791528418559343151187932905776427998012736052409 44\\ &611733704181896249474745967569047983987684014280497359001735474 13\\ &191162934865894639545423132081056990198071939175430299848814901 93\\ &193671823126420427276358911487848329996467356160858436515425171 82\\ &4179566414953524432921936494838577722534600 88 \end{aligned}$$

Substituting precise Riemann zeros into (19) one can check that the equation is identically satisfied. These results corroborate that (19) is an exact equation for the Riemann zeros, and we emphasize that it was derived on the critical line.

VII. NUMERICAL ANALYSIS: *L*-FUNCTIONS

We perform exactly the same numerical procedure as described in the previous section VI D, but now with equation (42) and (66) for Dirichlet L-functions, or with (58) and (67) for L-functions based on level one modular forms.

A. Dirichlet *L*-functions

We will illustrate our formulas with the primitive characters $\chi_{7,2}$ and $\chi_{7,3}$, since they possess the full generality of $a = 0$ and $a = 1$ and complex components. There are actually $\varphi(7) = 6$ distinct characters mod 7.

Example $\boldsymbol{\chi_{7,2}}$. Consider $k = 7$ and $j = 2$, i.e. we are computing the Dirichlet character $\chi_{7,2}(n)$. For this case $a = 1$. Then we have the following components:

n	1	2	3	4	5	6	7
$\chi_{7,2}(n)$	1	$e^{2\pi i/3}$	$e^{\pi i/3}$	$e^{-2\pi i/3}$	$e^{-\pi i/3}$	-1	0

(76)

The first few zeros, positive and negative, obtained by solving (42) are shown in TABLE VI (see Appendix B for the Mathematica implementation). The solutions shown are easily obtained with 50 decimal places of accuracy, and agree with the ones in [42], which were computed up to 20 decimal places.

Example $\boldsymbol{\chi_{7,3}}$. Consider $k = 7$ and $j = 3$, such that $a = 0$. In this case the components of $\chi_{7,3}(n)$ are the following:

n	1	2	3	4	5	6	7
$\chi_{7,3}(n)$	1	$e^{-2\pi i/3}$	$e^{2\pi i/3}$	$e^{2\pi i/3}$	$e^{-2\pi i/3}$	1	0

(77)

The first few solutions of (42) are shown in TABLE VII and are accurate up to 50 decimal places, and agree with the ones obtained in [42]. As stated previously, the solutions to equation (42) can be calculated to any desired level of accuracy. For instance, continuing

n	$\widetilde{y}_n$	y_n
10	25.57	25.68439458577475868571703403827676455384372032540097
9	23.67	24.15466453997877089700472248737944003578203821931614
8	21.73	21.65252506979642618329545373529843196334089625358303
7	19.73	19.65122423323359536954110529158230382437142654926200
6	17.66	17.16141654370607042290552256158565828745960439000612
5	15.50	15.74686940763941532761353888536874657958310887967059
4	13.24	13.85454287448149778875634224346689375234567535103602
3	10.81	9.97989590209139315060581291354262017420478655402522
2	8.14	8.41361099147117759845752355454727442365106861800819
1	4.97	5.19811619946654558608428407430395403442607551643259
0	−3.44	−2.50937455292911971967838452268365746558148671924805
−1	−7.04	−7.48493173971596112913314844807905530366284046079242
−2	−9.85	−9.89354379409772210349418069925221744973779313289503
−3	−12.35	−12.25742488648921665489461478678500208978360618268664
−4	−14.67	−14.13507775903777080989456447454654848575048882728616
−5	−16.86	−17.71409256153115895322699037454043289926793578042465
−6	−18.96	−18.88909760017588073794865307957219593848843485334695
−7	−20.99	−20.60481911491253262583427068994945289180639925014034
−8	−22.95	−22.66635642792466587252079667063882618974425685038326
−9	−24.87	−25.28550752850252321309973718800386160807733038068585

TABLE VI: Numerical solutions of (42) starting with the approximation (66), for the character (76). The solutions are accurate to 50 decimal places and verified to $\left|L\left(\frac{1}{2}+iy_n\right)\right| \sim 10^{-50}$.

with the character $\chi_{7,3}$, we can easily compute the following number for $n = 1000$, accurate to 100 decimal places, i.e. 104 digits:

$$
\begin{aligned}
y_{1000} = 1037.&56371706920654296560046127698168717112749601359549 \\
&01734503731679747841764715443496546207885576444206
\end{aligned}
$$

We also have been able to solve the equation for high zeros to high accuracy, up to the millionth zero, some of which are listed in TABLE VIII, and were previously unknown.

n	$\widetilde{y}_n$	y_n
10	25.55	26.16994490801983565967242517629313321888238615283992
9	23.65	23.20367246134665537826174805893362248072979160004334
8	21.71	21.31464724410425595182027902594093075251557654412326
7	19.71	20.03055898508203028994206564551578139558919887432101
6	17.64	17.61605319887654241030080166645399190430725521508443
5	15.48	15.93744820468795955688957399890407546316342953223035
4	13.21	12.53254782268627400807230480038783642378927939761728
3	10.79	10.73611998749339311587424153504894305046993275660967
2	8.11	8.78555471449907536558015746317619235911936921514074
1	4.93	4.35640162473628422727957479051551913297149929441224
0	−5.45	−6.20123004275588129466099054628663166500168462793701
−1	−8.53	−7.92743089809203774838798659746549239024181788857305
−2	−11.15	−11.01044486207249042239362741094860371668883190429106
−3	−13.55	−13.82986789986136757061236809479729216775842888684529
−4	−15.80	−16.01372713415040781987211528577709085306698639304444
−5	−17.94	−18.04485754217402476822077016067233558476519398664936
−6	−20.00	−19.11388571948958246184820859785760690560580302023623
−7	−22.00	−22.75640595577430793123629559665860790727892846161121
−8	−23.94	−23.95593843516797851393076448042024914372113079309104
−9	−25.83	−25.72310440610835748550521669187512401719774475488087

TABLE VII: Numerical solutions of (42) starting with the approximation (66), for the character (77). The solutions are accurate to 50 decimal places and verified to $\left|L\left(\frac{1}{2}+iy_n\right)\right| \sim 10^{-50}$.

n	$\widetilde{y}_n$	y_n
10^3	1037.61	1037.563717069206542965600461276981687171127496013595490
10^4	7787.18	7787.337916840954922060149425635486826208937584171726906
10^5	61951.04	61950.779420880674657842482173403370835983852937763461400
10^6	512684.78	512684.856698029779109684519709321053301710419463624401290

TABLE VIII: Higher zeros for the Dirichlet character (77). These solutions to (42) are accurate to 50 decimal places.

B. Modular L-function based on Ramanujan τ

Here we will consider an example of a modular form of weight $k = 12$. The simplest example is based on the Dedekind η-function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{78}$$

Up to a simple factor, η is the inverse of the chiral partition function of the free boson conformal field theory [44], where τ is the modular parameter of the torus. The modular discriminant

$$\Delta(\tau) = \eta(\tau)^{24} = \sum_{n=1}^{\infty} \tau(n)\, q^n \tag{79}$$

is a weight $k = 12$ modular form. It is closely related to the inverse of the partition function of the bosonic string in 26 dimensions, where 24 is the number of light-cone degrees of freedom [45]. The Fourier coefficients $\tau(n)$ correspond to the Ramanujan τ-function, and the first few are

n	1	2	3	4	5	6	7	8	9
$\tau(n)$	1	-24	252	-1472	4830	-6048	-16744	84480	-113643

(80)

We then define the Dirichlet series

$$L_\Delta(z) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^z}. \tag{81}$$

Applying (58), the zeros are $\rho_n = 6 + iy_n$, where the y_n satisfy the exact equation

$$\vartheta_{12}(y) + \lim_{\delta\to 0^+} \arg L_\Delta(6 + \delta + iy_n) = \left(n - \tfrac{1}{2}\right)\pi. \tag{82}$$

The counting function (60) and its asymptotic approximation are

$$N_0(T) = \frac{1}{\pi}\vartheta_{12}(T) + \frac{1}{\pi}\arg L_\Delta(6 + iT) \tag{83}$$

$$\approx \frac{T}{\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{1}{\pi}\arg L_\Delta(6 + iT) + \frac{11}{4}. \tag{84}$$

A plot of (83) is shown in FIG. 6, and we can see that it is a perfect staircase function. The approximate solution (67) now have the form

$$\widetilde{y}_n = \frac{\left(n - \frac{13}{4}\right)\pi}{W\left[(2e)^{-1}\left(n - \frac{13}{4}\right)\right]} \qquad (n = 2, 3, \dots). \tag{85}$$

Note that the above equation is valid for $n > 1$, since $W(x)$ is not defined for $x < -1/e$.

We follow exactly the same procedure, previously discussed in section VI D and implemented in Appendix B, to solve the equation (82) starting with the approximation provided by (85). Some of these solutions are shown in TABLE IX and are accurate to 50 decimal places.

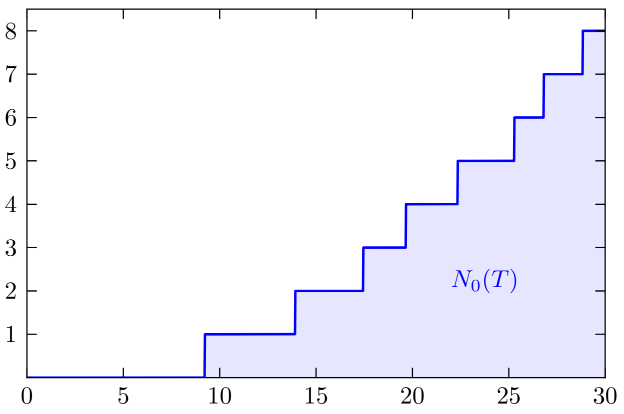

FIG. 6: Exact counting formula (83) based on the Ramanujan τ-function.

n	$\widetilde{y}_n$	y_n
1		9.22237939992110252224376719274347813552877062243201
2	12.46	13.90754986139213440644668132877021949175755235351449
3	16.27	17.44277697823447331355152513712726271870886652427527
4	19.30	19.65651314195496100012728175632130280161555091200324
5	21.94	22.33610363720986727568267445923624619245504695246527
6	24.35	25.27463654811236535674532419313346311859592673122941
7	26.60	26.80439115835040303257574923358456474715296800497933
8	28.72	28.83168262418687544502196191298438972569093668609124
9	30.74	31.17820949836025906449218889077405585464551198966267
10	32.68	32.77487538223120744183045567331198999909916163721260
100	143.03	143.08355526347845507373979776964664120256210342087127
200	235.55	235.74710143999213667703807130733621035921210614210694
300	318.61	318.36169446742310747533323741641236307865855919162340

TABLE IX: Non-trivial zeros of the modular L-function based on the Ramanujan τ-function, obtained from (82) starting with the approximation (85). These solutions are accurate to 50 decimal places.

VIII. CONCLUDING REMARKS

In this paper we considered non-trivial zeros of the Riemann ζ-function, Dirichlet L-functions and L-functions based on level one modular forms. The same approach was applied to all these classes of functions, showing that there are an infinite number of zeros on the critical line in one-to-one correspondence with the zeros of the cosine function (10), leading to a transcendental equation satisfied by the ordinate of the n-th zero. More specifically, for the Riemann ζ-function these zeros are solutions to (19), for Dirichlet L-functions we have (42), and for L-functions based on level one modular forms the ordinates of the zeros must satisfy (58). It is important to stress that these equations were *derived on the critical line*, without assuming the RH.

The implication of our work for the (generalized) RH can be summarized as follows. We have presented arguments that for each of these classes of functions, there is a unique solution to the transcendental equation we derived for the n-th zero. If this is indeed the case, then the zeros we have found on the critical line can be counted, since they are enumerated by the integer n. The zeros we have found on the critical line saturate the known counting formulas for the zeros on the entire critical strip, and the RH follows.

We also showed that it is possible to obtain an explicit formula as an approximation for the ordinates of the zeros; equation (64) for the ζ-function, (66) for Dirichlet L-functions and (67) for L-functions based on level one modular forms. This approximation is very convenient, allowing us to actually compute accurate zeros without relying on Gram points, nor dealing with violations of Gram's law.

We have also provided strong numerical evidence for the validity of these equations satisfied by the n-th zero. For the ζ-function, the leading order asymptotic approximation (12) proved to be accurate enough to reveal the interesting features of the Riemann zeros, like the GUE statistics and the reconstruction of the prime number counting function. Under the numerical analysis, the exact equation (19) is much more stable and is identically satisfied by the Riemann zeros; it is thus able to provide numerical results as accurate as desirable. We have also provided accurate numerical solutions for Dirichlet L-functions based on (42) and for the particular example of the modular L-function based on the Ramanujan τ-function, based on (82). The numerical approach employed here constitutes a novel and simple method to compute the zeros.

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Appendix A: General L-functions

With the aim of unifying the previous results, we extend the main equations to a more general, but not specific, class of L-functions. We must of course assume that the analysis in sections II and III is valid, and this must be checked case by case. Thus one must bear in mind that since we are not specific about the L-function, but just assume some elementary properties, there is no guarantee that the latter analysis is valid for every L-function with the properties below. We are thus simply going to assume that the previous analysis is valid and present the resulting equations that would follow.

We are going to consider L-functions with the properties outlined in [26, Chap. 5]. The general L-function has a Dirichlet series

$$L(z, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^z}, \tag{A1}$$

where $\lambda_f(1) = 1$ and $\lambda_f(n)$ is a complex number. This series is convergent for $\Re(z) > 1$, it has an Euler product of degree $d \geq 1$, where d is an integer, and admits an analytic continuation in the whole complex plane, except for poles at $z = 0$ and $z = 1$. The arithmetic object f has no specific meaning here. For instance, f can be a modular or cusp form, or it can be associated to Dirichlet characters $\lambda_f(n) = \chi(n)$. The object f defines the particular class of L-functions. Let us introduce

$$\gamma\left(z, f\right) = \pi^{-dz/2} \prod_{j=1}^{d} \Gamma\left(\frac{z + a_j}{2}\right), \tag{A2}$$

where the complex numbers a_j, which come in conjugate pairs, are the so called local parameters at infinity. Let us define

$$\Lambda\left(z, f\right) \equiv k^{z/2} \gamma(z, f)\, L(z, f), \tag{A3}$$

where $k = k(f) \geq 1$ is an integer, the *conductor* of $L(z, f)$. Then it satisfies the functional equation [26]

$$\Lambda(z, f) = \alpha(f) \Lambda^*(1 - z, f) \equiv \alpha(f) \left(\Lambda(1 - z^*, f)\right)^*. \tag{A4}$$

Here α is a complex phase, i.e. $|\alpha| = 1$. The symbol f^* denotes the dual of f, associated to the Dirichlet series with $\lambda_{f^*}(n) = \left(\lambda_f(n)\right)^*$. We also have the relations $\gamma(z, f^*) = \gamma(z, f)$ and $k(f^*) = k(f)$.

We can write the functional equation in a more symmetric form by introducing

$$\xi(z,f) \equiv e^{-i\beta/2}\Lambda(z,f), \tag{A5}$$

where $\alpha(f) = e^{i\beta(f)}$. Then we have the functional equation

$$\xi(z,f) = \xi^*(1-z,f) = \left(\xi(1-z^*,f)\right)^*. \tag{A6}$$

Writing the polar form $\xi(z,f) = A(x,y,f)e^{i\theta(x,y,f)}$ we have

$$A(x,y,f) = k^{x/2}\pi^{-dx/2}\left|L(x+iy,f)\right|\prod_{j=1}^{d}\left|\Gamma\left(\frac{x+a_j+iy}{2}\right)\right|, \tag{A7}$$

$$\theta(x,y,f) = \sum_{j=1}^{d}\arg\Gamma\left(\frac{x+a_j+iy}{2}\right) + \arg L(x+iy,f) + \frac{y}{2}\log\left(\frac{k}{\pi^d}\right) - \frac{\beta}{2}. \tag{A8}$$

Denoting $\xi^*(1-z,f) = A'(x,y,f)e^{-i\theta'(x,y,f)}$, we then have $A'(x,y,f) = A(1-x,y,f)$ and $\theta'(x,y,f) = \theta(1-x,y,f)$. Let us define the generalized Riemann-Siegel ϑ_{k,a_j} function

$$\vartheta_{k,a_j}(y) \equiv \arg\Gamma\left(\frac{1}{4}+\frac{a_j}{2}+i\frac{y}{2}\right) - \frac{y}{2}\log\left(\frac{\pi}{k^{1/d}}\right). \tag{A9}$$

Following the same previous analysis, by imposing the identity

$$\lim_{\delta\to 0^+}\left\{\xi(\rho+\delta,f) + \xi^*(1-\rho-\delta,f)\right\} = 0 \tag{A10}$$

where $\rho = x+iy$ is a non-trivial L-zero, we take the particular solution $\theta = \theta'$ and $\lim_{\delta\to 0^+}\cos\theta = 0$. Thus the first equation is automatically satisfied on the critical line $\Re(\rho) = \frac{1}{2}$, and then $\lim_{\delta\to 0^+}\theta\left(\frac{1}{2}+\delta,y\right) = \left(n+\frac{1}{2}\right)\pi$ solves the second equation, yielding the equation for the n-th zero. Introducing a shift $n \to n-(n_0+1)$, where n_0 should be determined by each specific case according to the convention that the first positive zero is labelled by $n=1$ (we will omit n_0 in the following), we then conclude that these zeros have the form $\rho_n = \frac{1}{2}+iy_n$ for $n\in\mathbb{Z}$, and y_n is the solution of the equation

$$\sum_{j=1}^{d}\vartheta_{k,a_j}(y_n) + \lim_{\delta\to 0^+}\arg L\left(\tfrac{1}{2}+\delta+iy_n,f\right) - \frac{\beta}{2} = \left(n-\tfrac{1}{2}\right)\pi. \tag{A11}$$

It is also possible to replace β by noting that $\xi(\frac{1}{2}+iy,f)$ is real, thus $\arg\xi\left(\frac{1}{2}\right)=0$ then

$$\frac{\beta(f)}{2} = \sum_{j=1}^{d}\arg\Gamma\left(\frac{1}{4}+\frac{a_j}{2}\right) + \arg L\left(\tfrac{1}{2},f\right). \tag{A12}$$

For real a_j the first term vanishes. If $f^* = f$ then $L(z, f)$ is said to be self-dual. If besides this $\alpha(f) = -1$, then $L\left(\frac{1}{2}, f\right) = 0$.

The counting formula on the critical line, for $0 < \Im(\rho) < T$, can be obtained from (A11) by replacing $y_n \to T$ and $n \to N_0^+(T, f) + \frac{1}{2}$, thus

$$N_0^+(T, f) = \frac{1}{\pi}\sum_{j=1}^{d}\left\{\vartheta_{k,a_j}(T) - \arg\Gamma\left(\frac{1}{4} + \frac{a_j}{2}\right)\right\} + \frac{1}{\pi}\arg L\left(\tfrac{1}{2} + iT, f\right) - \frac{1}{\pi}\arg L\left(\tfrac{1}{2}, f\right). \tag{A13}$$

As discussed previously in section II, this is valid if (A11) is well defined for any y and has a unique solution for every n. The same counting on the whole strip $N^+(T, f)$ can be obtained through the standard Cauchy's argument principle [26]. Thus $N_0^+(T, f) = N^+(T, f)$, justifying that the particular solution captures all non-trivial zeros on the critical strip, therefore they must be all on the critical line. The counting on the negative half line $-T < \Im(\rho) < 0$ can be obtained from $N_0^-(T, f) = N_0^+(T, f^*)$. Expanding (A9) from Stirling's formula we have

$$\vartheta_{k,a_j}(y) = \frac{y}{2}\log\left(\frac{k^{1/d}\,y}{2\pi e}\right) + \frac{2a_j - 1}{8}\pi + O(1/y). \tag{A14}$$

Then from (A13) we have

$$N_0^+(T, f) = \frac{T}{2\pi}\log\left(\frac{k\,T^d}{(2\pi e)^d}\right) + \sum_{j=1}^{d}\left\{\frac{2a_j - 1}{8} - \frac{1}{\pi}\arg\Gamma\left(\frac{1}{4} + \frac{a_j}{2}\right)\right\}$$
$$+ \frac{1}{\pi}\arg L\left(\tfrac{1}{2} + iT, f\right) - \frac{1}{\pi}L\left(\tfrac{1}{2}, f\right) + O(1/y). \tag{A15}$$

Using (A14) in (A11) and neglecting the small $\arg L\left(\frac{1}{2} + iT, f\right)$ term, it is possible to obtain an approximate solution in closed form, which reads

$$\widetilde{y}_n = \frac{2\pi A_n}{d\,W\left[k^{1/d}(d\,e)^{-1}A_n\right]}, \tag{A16}$$

where $W(x)$ is the principal value of the Lambert function over real values, and

$$A_n = n - \frac{1}{2} + \frac{1}{\pi}\arg L\left(\tfrac{1}{2}, f\right) + \sum_{j=1}^{d}\left\{\frac{1}{\pi}\arg\Gamma\left(\frac{1}{4} + \frac{a_j}{2}\right) - \frac{2a_j - 1}{8}\right\}. \tag{A17}$$

Appendix B: Mathematica code

Here we provide the short Mathematica code used to compute the zeros from the transcendental equations. We will consider Dirichlet L-functions, since it involves more ingredients, like the modulus k, the order a and the Gauss sum $G(\tau)$. For the Riemann ζ-function the procedure is trivially adapted, as well as for the Ramanujan τ-function of section VII B.

The function (39) is implemented as follows:

```
RSTheta[y_, a_, k_] := Im[LogGamma[1/4 + a/2 + I*y/2]] − y/2*Log[Pi/k]
```

For the transcendental equation (42) we have

```
ExactEq[n_, y_, s_, a_, k_, j_, G_, n0_] :=
    (RSTheta[y, a, k] + Arg[DirichletL[k, j, 1/2 + s + I*y]] −1/2*Arg[G])/Pi + a/4 + 1/2 − n + n0
```

Above, s denotes $0 < \delta \ll 1$, a is the order (27), k is the modulus, j specify the Dirichlet character $\chi_{k,j}$ (as discussed in section III), and G is the Gauss sum (24). Note that we also included n_0, discussed after (41), but we always set $n_0 = 0$ for the cases analysed in section VII A. The implementation of the approximate solution (66) is

```
Sgn[n_] := Which[n != 0, Sign[n], n == 0, −1]
A[n_, a_, G_, n0_] := Sgn[n]*(n − n0 + 1/2/Pi*Arg[G]) + (1 − 4*Sgn[n] − 2*a*(Sgn[n] + 1))/8
yApprox[n_, a_, G_, k_, n0_] := 2*Pi*Sgn[n]*A[n, a, G, n0]/LambertW[k*A[n, a, G, n0]/E]
```

One can then obtain the numerical solution of the transcendental equation (42) as follows:

```
FindZero[n_, s_, a_, k_, j_, G_, n0_, y0_, prec_] :=
 y /. FindRoot[ExactEq[n, y, s, a, k, j, G, n0], {y, y0}, PrecisionGoal −> prec/2, AccuracyGoal
     −> prec/2, WorkingPrecision −> prec]
```

Above, y_0 will be given by the approximate solution (66). The variable *prec* will be adjusted iteratively. Now the procedure described in section VI D can be implemented as follows:

```
DirichletNZero[n_, order_, digits_, k_, j_, n0_] := (
    chi = DirichletCharacter[k, j, −1];
    a = Which[chi == −1, 1, chi == 1, 0];
    s = 10ˆ(−3);
    prec = 15;
    G = Sum[DirichletCharacter[k, j, l]*Exp[2*Pi*I*l/k], {l, 1, k}];
    y = N[yApprox[n, a, G, k, n0], 20];
    absvalue = 1;
```

```
    While[absvalue > order,
        y = FindZero[n, s, a, k, j, G, n0, y, prec];
        Print[NumberForm[y, digits]];
        s = s/1000;
        prec = prec + 20;
        absvalue = Abs[DirichletL[k, j, 1/2 + I*y]];
    ]
    Print[ScientificForm[absvalue, 5]];
)
```

Above, the variable *order* controls the accuracy of the solution. For instance, if *order* = 10^{-50}, it will iterate until the solution is verified at least to $|L\left(\frac{1}{2}\right) + iy\big)| \sim 10^{-50}$. The variable *digits* controls the number of decimal places shown in the output.

Let us compute the zero $n = 1$, for the character (77), i.e. $k = 7$ and $j = 3$. We will verify the solution to *order* $= 10^{-50}$ and print the results with *digits* $= 52$. Thus executing

```
DirichletNZero[1, 10^(−50), 52, 7, 3, 0]
```

the output will be

```
4.35640188194945
4.356401624736541498075485169l149585
4.3564016247362842275368456440147597631978362633512 53
```

Note how the decimal digits converge in each iteration. By decreasing *order* and increasing *digits* it is possible to obtain highly accurate solutions. It is exactly in this way that we obtained the tables shown in section VII A. Obviously, depending on the height of the critical line under consideration, one should adapt the parameters s and $prec$ appropriately. In Mathematica we were able to compute solutions up to $n \sim 10^6$ for Dirchlet L-functions, and up to $n \sim 10^9$ for the Riemann ζ-function without problems. We were unable to go much higher only because Mathematica could not compute the arg term reliably. To solve the transcendental equations (19) and (42) for very high values on the critical line is a challenging numerical problem. Nevertheless, we believe that it can be done through a more specialized implementation.

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[46] The linear combination in (8) was chosen to be manifestly symmetric under $z \to 1 - z$. Had we taken a different linear combination in (8), then $B = e^{i\theta} + b\, e^{-i\theta'}$ for some constant b. Setting the real and imaginary parts of B to zero gives the two equations $\cos\theta + b\cos\theta' = 0$ and $\sin\theta - b\sin\theta' = 0$. Summing the squares of these equations one obtains $\cos(\theta + \theta') = -(b + 1/b)/2$. However, since $b + 1/b > 1$, there are no solutions except for $b = 1$.

[47] Our enumeration convention for the j-index of $\chi_{k,j}$ is taken from Mathematica.

[48] Some care must be taken in numerically evaluating $\mathrm{Li}(x^\rho)$ since Li has a branch point. It is more properly defined as $\mathrm{Ei}(\rho \log x)$ where $\mathrm{Ei}(z) = -\int_{-z}^{\infty} dt\, e^{-t}/t$ is the exponential integral function.

[49] Computing this number to 500 digit accuracy took a few minutes on a standard (8 GB RAM) personal laptop computer using Mathematica. It only takes a few seconds to obtain 100 digit accuracy.