Special series related to Lambert $\boldsymbol{W}$

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Abstract

First, we use the Lambert $W(z)$ function to analyse the n-iterates

$$f^{<n>}(z) = \underbrace{f \circ f \circ \cdots \circ f}_{n}(z) \tag{1}$$

of normalized formal power series of the form $f(z) = z(1 + a_1 z + a_2 z^2 + \cdots) \in \mathbb{C}[[z]]$. It is well known that the set $\mathbb{S}$ of such series form a (noncommutative) group under substitution in which the inverse of $f(z)$ is the "reverse" series $f^{<-1>}(z)$ which can be obtained by Lagrange inversion, for example. Specifically, we extend the "discrete" n-iterates $f^{<n>}(z)$, $n \in \mathbb{Z}$, to "continuous" s-iterates $f^{<s>}(z)$, $s \in \mathbb{C}$, and show that the computation of the s-iterates $f^{<s>}(z)$ of *any* $f(z) \in \mathbb{S}$ can be reduced to the computation of the s-iterates $u_{p,q}^{<s>}(z) \in \mathbb{S}$, of "universal" series, $u_{p,q}(z) \in \mathbb{S}$, where the parameters p and q depend only on the coefficients $a_1, a_2, \ldots$ appearing in $f(z)$. These iterates $u_{p,q}^{<s>}(z)$ have an *explicit analytic form* involving the Lambert W function.

Secondly, the Lambert $W(z)$ function is used as a "building block" for the construction of extensions of *itself* to k variables, $k = 2, 3, \ldots$. These extensions, which we denote by $W(z_1, z_2, \ldots, z_k)$, are naturally connected with classes of rooted trees that are invariant under the action of certain types of permutations applied to their nodes. For example, for $k = 2$, it turns out that

$$W(z_1, z_2) = W\left(z_1 \left(\frac{W(z_2)}{z_2}\right)^{\frac{1}{2}}\right), \tag{2}$$

where the principal branch of the square root is taken. The coefficients of the power series of $W(z_1, z_2)$ at $(0, 0)$ count rooted trees that are invariant under *involutions* applied to their nodes. Explicit formulas for the coefficients of the series of $W(z_1, z_2, \ldots, z_k)$ are also given.

Finally, we introduce some variants of the W-function in the context of "combinatorial power series". These are formal power series, in the indeterminate X, of the form

$$F(X) = \sum_{n,H} c_{n,H} \frac{X^n}{H}, \tag{3}$$

in which the coefficients $c_{n,H}$ are complex numbers and, for each n, H runs through the subgroups of the symmetric group S_n. Such series conveniently encode classes (or species) of combinatorial structures according to the automorphisms groups H of their structures. The usual operations between classical power series (including substitution and derivation) have been extended to this kind of series. In particular, we introduce a generalization $\mathcal{W}(X)$ of $W(X)$ through the "standard" formula $\mathcal{W}(X) = -T(-X)$ where $T(X)$ is the combinatorial power series (***not*** the classical power series) associated to the class T of rooted trees. Other variants of W arising from differential equations in the context of combinatorial power series are also presented.

Keywords

Lambert W function, nonlinear iterative schemes, rooted trees, species of structures