

Inverse Trig Function Integration Problem 1

$$\int \text{ArcSec}[a + b x] dx$$

- *Rubi* uses the substitution $u=a+bx$ to generalize rule:

`Int[ArcSec[x], x]`

$$x \text{ArcSec}[x] - \text{ArcTanh}\left[\sqrt{1 - \frac{1}{x^2}}\right]$$

`Int[ArcSec[a + b x], x]`

$$\frac{(a + b x) \text{ArcSec}[a + b x]}{b} - \frac{\text{ArcTanh}\left[\sqrt{1 - \frac{1}{(a+bx)^2}}\right]}{b}$$

- *Mathematica* does *not* use the substitution $u=a+bx$ to generalize rule:

`Int[ArcSec[x] dx]`

$$x \text{ArcSec}[x] - \frac{\sqrt{-1+x^2} \text{Log}\left[2\left(x + \sqrt{-1+x^2}\right)\right]}{\sqrt{1 - \frac{1}{x^2}}}$$

`Int[ArcSec[a + b x] dx]`

$$x \text{ArcSec}[a + b x] - \frac{(a + b x) \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}} \left(a \text{ArcTan}\left[\frac{1}{\sqrt{-1+a^2+2abx+b^2x^2}}\right] + \text{Log}\left[2\left(a + b x + \sqrt{-1+a^2+2abx+b^2x^2}\right)\right] \right)}{b \sqrt{-1+a^2+2abx+b^2x^2}}$$

- *Maple* uses the substitution $u=a+bx$ to generalize rule:

`int(arcsec(x), x);`

$$x \text{ArcSec}[x] - \text{Log}\left[x + \sqrt{1 - \frac{1}{x^2}}\right]$$

`int(arcsec(a + b * x), x);`

$$\frac{a \text{ArcSec}[a + b x]}{b} + x \text{ArcSec}[a + b x] - \frac{\text{Log}\left[a + b x + (a + b x) \sqrt{1 - \frac{1}{(a+bx)^2}}\right]}{b}$$

Note that these systems give similar results to the above for the arccosecant function.

Inverse Trig Function Integration Problem 2

$$\int \frac{\text{ArcSec}[a x^n]}{x} dx$$

- *Rubi* knows and takes advantage of the general rule for arbitrary n:

$$\text{Int}\left[\frac{\text{ArcSec}[a x^n]}{x}, x\right]$$

$$\frac{i \text{ArcSec}[a x^n]^2}{2n} - \frac{\text{ArcSec}[a x^n] \text{Log}\left[1 - \frac{1}{\left(\frac{i x^n}{a} + \sqrt{1 - \frac{x^{-2n}}{a^2}}\right)^2}\right]}{n} + \frac{i \text{PolyLog}\left[2, \frac{1}{\left(\frac{i x^n}{a} + \sqrt{1 - \frac{x^{-2n}}{a^2}}\right)^2}\right]}{2n}$$

$$\text{Int}\left[\frac{\text{ArcSec}[a x^5]}{x}, x\right]$$

$$\frac{1}{10} i \text{ArcSec}[a x^5]^2 - \frac{1}{5} \text{ArcSec}[a x^5] \text{Log}\left[1 - \frac{1}{\left(\sqrt{1 - \frac{1}{a^2 x^{10}} + \frac{i}{a x^5}}\right)^2}\right] + \frac{1}{10} i \text{PolyLog}\left[2, \frac{1}{\left(\sqrt{1 - \frac{1}{a^2 x^{10}} + \frac{i}{a x^5}}\right)^2}\right]$$

- *Mathematica* does not know the elementary form of the general rule, but returns an elementary form when n is numeric:

$$\int \frac{\text{ArcSec}[a x^n]}{x} dx$$

$$\frac{x^{-n} \text{HypergeometricPFQ}\left[\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, \frac{x^{-2n}}{a^2}\right]}{a n} + \left(\text{ArcSec}[a x^n] + \text{ArcSin}\left[\frac{x^{-n}}{a}\right]\right) \text{Log}[x]$$

$$\int \frac{\text{ArcSec}[a x^5]}{x} dx$$

$$\frac{1}{10} i \text{ArcSec}[a x^5]^2 - \frac{1}{5} \text{ArcSec}[a x^5] \text{Log}\left[1 + e^{2 i \text{ArcSec}[a x^5]}\right] + \frac{1}{10} i \text{PolyLog}\left[2, -e^{2 i \text{ArcSec}[a x^5]}\right]$$

- *Maple* knows the general rule for arbitrary n, but does not use it when n is numeric:

$$\text{int}(\text{arcsec}(a * x^n) / x, x);$$

$$\frac{i \text{ArcSec}[a x^n]^2}{2n} - \frac{\text{ArcSec}[a x^n] \text{Log}\left[1 + \left(\frac{x^{-n}}{a} + i \sqrt{1 - \frac{x^{-2n}}{a^2}}\right)^2\right]}{n} + \frac{i \text{PolyLog}\left[2, -\left(\frac{x^{-n}}{a} + i \sqrt{1 - \frac{x^{-2n}}{a^2}}\right)^2\right]}{2n}$$

$$\text{int}(\text{arcsec}(a * x^5) / x, x);$$

$$\int \frac{\text{ArcSec}[a x^5]}{x} dx$$

Note that these systems give similar results to the above for the arccosecant function.

Inverse Trig Function Integration Problem 3

$$\int \frac{\text{ArcTan}[x]}{(a + b x^2)^{m/2}} dx$$

- *Rubi* returns a relatively simple sum of $(m+1)/2$ terms for odd $m > 1$:

$$\text{Int}\left[\frac{\text{ArcTan}[x]}{(a + b x^2)^{3/2}}, x\right]$$

$$\frac{x \text{ArcTan}[x]}{a \sqrt{a + b x^2}} + \frac{\text{ArcTanh}\left[\frac{\sqrt{a + b x^2}}{\sqrt{a - b}}\right]}{a \sqrt{a - b}}$$

$$\text{Int}\left[\frac{\text{ArcTan}[x]}{(a + b x^2)^{5/2}}, x\right]$$

$$-\frac{1}{3 a (a - b) \sqrt{a + b x^2}} + \frac{x (3 a + 2 b x^2) \text{ArcTan}[x]}{3 a^2 (a + b x^2)^{3/2}} + \frac{(3 a - 2 b) \text{ArcTanh}\left[\frac{\sqrt{a + b x^2}}{\sqrt{a - b}}\right]}{3 a^2 (a - b)^{3/2}}$$

- *Mathematica* returns more complicated sums involving the imaginary unit:

$$\int \frac{\text{ArcTan}[x]}{(a + b x^2)^{3/2}} dx$$

$$\frac{x \text{ArcTan}[x]}{a \sqrt{a + b x^2}} + \frac{\text{Log}\left[-\frac{4 a (a - i b x + \sqrt{a - b} \sqrt{a + b x^2})}{\sqrt{a - b} (i + x)}\right]}{2 a \sqrt{a - b}} + \frac{\text{Log}\left[-\frac{4 a (a + i b x + \sqrt{a - b} \sqrt{a + b x^2})}{\sqrt{a - b} (-i + x)}\right]}{2 a \sqrt{a - b}}$$

$$\int \frac{\text{ArcTan}[x]}{(a + b x^2)^{5/2}} dx$$

$$-\frac{1}{3 a (a - b) \sqrt{a + b x^2}} + \frac{x (3 a + 2 b x^2) \text{ArcTan}[x]}{3 a^2 (a + b x^2)^{3/2}} + \frac{(3 a - 2 b) \text{Log}\left[-\frac{12 a^2 \sqrt{a - b} (a - i b x + \sqrt{a - b} \sqrt{a + b x^2})}{(3 a - 2 b) (i + x)}\right]}{6 a^2 (a - b)^{3/2}} + \frac{(3 a - 2 b) \text{Log}\left[-\frac{12 a^2 \sqrt{a - b} (a + i b x + \sqrt{a - b} \sqrt{a + b x^2})}{(3 a - 2 b) (-i + x)}\right]}{6 a^2 (a - b)^{3/2}}$$

- *Maple* is unable to integrate the expressions:

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int (arctan (x) / (a + b * x^2) ^ (3 / 2), x);
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$$\int \frac{\text{ArcTan}[x]}{(a + b x^2)^{3/2}} dx$$

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int (arctan (x) / (a + b * x^2) ^ (5 / 2), x);
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$$\int \frac{\text{ArcTan}[x]}{(a + b x^2)^{5/2}} dx$$

Note that these systems give similar results to the above for the arccotangent, hyperbolic arctangent and

hyperbolic arccotangent functions.

Inverse Trig Function Integration Problem 4

$$\int \text{ArcTan}[e^{a+bx}] dx$$

- The *Rubi* result is a relatively simple 2 term sum:

$$\text{Int}[\text{ArcTan}[e^{a+bx}], x]$$

$$\frac{i \text{ PolyLog}[2, -i e^{a+bx}]}{2 b} - \frac{i \text{ PolyLog}[2, i e^{a+bx}]}{2 b}$$

- The *Mathematica* result is a 5 term sum, the first 3 of which are superfluous since their derivative is 0:

$$\int \text{ArcTan}[e^{a+bx}] dx$$

$$x \text{ ArcTan}[e^{a+bx}] - \frac{1}{2} i x \text{ Log}[1 - i e^{a+bx}] + \frac{1}{2} i x \text{ Log}[1 + i e^{a+bx}] + \frac{i \text{ PolyLog}[2, -i e^{a+bx}]}{2 b} - \frac{i \text{ PolyLog}[2, i e^{a+bx}]}{2 b}$$

- The *Maple* result is a complicated 6 term sum:

$$\text{int}(\arctan(\exp(a + b * x)), x);$$

$$\begin{aligned} & - \frac{\text{ArcTan}[e^{a+bx}] \text{Log}\left[\frac{2i}{i+e^{a+bx}}\right]}{b} + \frac{\text{ArcTan}[e^{a+bx}] \text{Log}\left[1 - \frac{1+i e^{a+bx}}{\sqrt{1+e^{2(a+bx)}}}\right]}{b} + \frac{\text{ArcTan}[e^{a+bx}] \text{Log}\left[1 + \frac{1+i e^{a+bx}}{\sqrt{1+e^{2(a+bx)}}}\right]}{b} + \\ & \frac{i \text{ PolyLog}\left[2, \frac{-i+e^{a+bx}}{i+e^{a+bx}}\right]}{2 b} - \frac{i \text{ PolyLog}\left[2, -\frac{1+i e^{a+bx}}{\sqrt{1+e^{2(a+bx)}}}\right]}{b} - \frac{i \text{ PolyLog}\left[2, \frac{1+i e^{a+bx}}{\sqrt{1+e^{2(a+bx)}}}\right]}{b} \end{aligned}$$

Note that these systems give similar results to the above for the arccotangent function.