

## Rules for integrands involving gamma functions

$$1. \int u \text{Gamma}[n, a + b x] dx$$

$$1: \int \text{Gamma}[n, a + b x] dx$$

Derivation: Integration by parts

$$\text{Basis: } \partial_x \text{Gamma}[n, a + b x] = - \frac{b (a + b x)^{n-1}}{e^{a + b x}}$$

Rule:

$$\int \text{Gamma}[n, a + b x] dx \rightarrow \frac{(a + b x) \text{Gamma}[n, a + b x]}{b} + \int \frac{(a + b x)^n}{e^{a + b x}} dx \rightarrow \frac{(a + b x) \text{Gamma}[n, a + b x]}{b} - \frac{\text{Gamma}[n + 1, a + b x]}{b}$$

Program code:

```
Int[Gamma[n_, a_. + b_. * x_], x_Symbol] :=
(a + b * x) * Gamma[n, a + b * x] / b - Gamma[n + 1, a + b * x] / b /;
FreeQ[{a, b, n}, x]
```

$$2. \int (dx)^m \text{Gamma}[n, b x] dx$$

$$1. \int \frac{\text{Gamma}[n, b x]}{x} dx$$

$$1. \int \frac{\text{Gamma}[n, b x]}{x} dx \text{ when } n \in \mathbb{Z}$$

$$1: \int \frac{\text{Gamma}[0, b x]}{x} dx$$

Basis:  $\text{Gamma}[0, z] = \text{ExpIntegralE}[1, z]$

Rule:

$$\int \frac{\text{Gamma}[0, b x]}{x} dx \rightarrow \int \frac{\text{ExpIntegralE}[1, b x]}{x} dx$$

$$\rightarrow b x \text{HypergeometricPFQ}[\{1, 1, 1\}, \{2, 2, 2\}, -b x] - \text{EulerGamma} \text{Log}[x] - \frac{1}{2} \text{Log}[b x]^2$$

### Program code:

```
Int[Gamma[0, b_. * x_] / x_, x_Symbol] :=
  b * x * HypergeometricPFQ[{1, 1, 1}, {2, 2, 2}, -b * x] - EulerGamma * Log[x] - 1/2 * Log[b * x]^2 /;
FreeQ[b, x]
```

$$x: \int \frac{\text{Gamma}[1, b x]}{x} dx$$

### Derivation: Algebraic expansion

Basis:  $\text{Gamma}[1, z] = \frac{1}{e^z}$

Note: *Mathematica* automatically evaluates  $\text{Gamma}[1, z]$  to  $e^{-z}$ .

Rule: If  $n > 1$ , then

$$\int \frac{\text{Gamma}[1, b x]}{x} dx \rightarrow \int \frac{1}{x e^{b x}} dx$$

### Program code:

```
(* Int[Gamma[1, b_. * x_] / x_, x_Symbol] :=
  Int[1 / (x * E^(b * x)), x] /;
FreeQ[b, x] *)
```

$$2: \int \frac{\text{Gamma}[n, b x]}{x} dx \text{ when } n - 1 \in \mathbb{Z}^+$$

Derivation: Algebraic expansion

$$\text{Basis: } \text{Gamma}[n, z] = \frac{z^{n-1}}{e^z} + (n-1) \text{Gamma}[n-1, z]$$

Rule: If  $n - 1 \in \mathbb{Z}^+$ , then

$$\int \frac{\text{Gamma}[n, b x]}{x} dx \rightarrow b \int \frac{(b x)^{n-2}}{e^{b x}} dx + (n-1) \int \frac{\text{Gamma}[n-1, b x]}{x} dx \rightarrow -\text{Gamma}[n-1, b x] + (n-1) \int \frac{\text{Gamma}[n-1, b x]}{x} dx$$

Program code:

```
Int[Gamma[n_,b_.*x_]/x_,x_Symbol] :=
  -Gamma[n-1,b*x] + (n-1)*Int[Gamma[n-1,b*x]/x,x] /;
FreeQ[b,x] && IGtQ[n,1]
```

$$3: \int \frac{\text{Gamma}[n, b x]}{x} dx \text{ when } n \in \mathbb{Z}^-$$

Derivation: Algebraic expansion

$$\text{Basis: } \text{Gamma}[n, z] = -\frac{z^n}{n e^z} + \frac{1}{n} \text{Gamma}[n+1, z]$$

Rule: If  $n \in \mathbb{Z}^-$ , then

$$\int \frac{\text{Gamma}[n, b x]}{x} dx \rightarrow -\frac{b}{n} \int \frac{(b x)^{n-1}}{e^{b x}} dx + \frac{1}{n} \int \frac{\text{Gamma}[n+1, b x]}{x} dx \rightarrow \frac{\text{Gamma}[n, b x]}{n} + \frac{1}{n} \int \frac{\text{Gamma}[n+1, b x]}{x} dx$$

Program code:

```
Int[Gamma[n_,b_.*x_]/x_,x_Symbol] :=
  Gamma[n,b*x]/n + 1/n*Int[Gamma[n+1,b*x]/x,x] /;
FreeQ[b,x] && ILtQ[n,0]
```

2:  $\int \frac{\text{Gamma}[n, b x]}{x} dx$  when  $n \notin \mathbb{Z}$

Rule: If  $n \notin \mathbb{Z}$ , then

$$\int \frac{\text{Gamma}[n, b x]}{x} dx \rightarrow \text{Gamma}[n] \text{Log}[x] - \frac{(b x)^n}{n^2} \text{HypergeometricPFQ}[\{n, n\}, \{1+n, 1+n\}, -b x]$$

Program code:

```
Int[Gamma[n_, b_. * x_] / x_, x_Symbol] :=
  Gamma[n] * Log[x] - (b * x)^n / n^2 * HypergeometricPFQ[{n, n}, {1+n, 1+n}, -b * x] /;
  FreeQ[{b, n}, x] && Not[IntegerQ[n]]
```

$$2: \int (d x)^m \text{Gamma}[n, b x] dx \text{ when } m \neq -1$$

Derivation: Integration by parts and piecewise constant extraction

$$\text{Basis: } \partial_x \frac{(d x)^m}{(b x)^m} == 0$$

$$\text{Basis: } -\frac{1}{b} \partial_x \text{Gamma}[m + n + 1, b x] == \frac{(b x)^{m+n}}{e^{b x}}$$

Note: The antiderivative is given directly without recursion so it is expressed entirely in terms of the incomplete gamma function without need for the exponential function.

Rule: If  $m \neq -1$ , then

$$\begin{aligned} \int (d x)^m \text{Gamma}[n, b x] dx &\rightarrow \frac{(d x)^{m+1} \text{Gamma}[n, b x]}{d (m+1)} + \frac{1}{m+1} \int \frac{(d x)^m (b x)^n}{e^{b x}} dx \\ &\rightarrow \frac{(d x)^{m+1} \text{Gamma}[n, b x]}{d (m+1)} + \frac{(d x)^m}{(m+1) (b x)^m} \int \frac{(b x)^{m+n}}{e^{b x}} dx \\ &\rightarrow \frac{(d x)^{m+1} \text{Gamma}[n, b x]}{d (m+1)} - \frac{(d x)^m \text{Gamma}[m+n+1, b x]}{b (m+1) (b x)^m} \end{aligned}$$

Program code:

```
Int[(d_*x_)^m_*Gamma[n_,b_*x_],x_Symbol] :=
  (d*x)^(m+1)*Gamma[n,b*x]/(d*(m+1)) -
  (d*x)^m*Gamma[m+n+1,b*x]/(b*(m+1)*(b*x)^m) /;
FreeQ[{b,d,m,n},x] && NeQ[m,-1]
```

$$3. \int (c + d x)^m \text{Gamma}[n, a + b x] dx$$

$$1: \int (c + d x)^m \text{Gamma}[n, a + b x] dx \text{ when } b c - a d = 0$$

Derivation: Integration by substitution

Rule: If  $b c - a d = 0$ , then

$$\int (c + d x)^m \text{Gamma}[n, a + b x] dx \rightarrow \frac{1}{b} \text{Subst} \left[ \int \left( \frac{d x}{b} \right)^m \text{Gamma}[n, x] dx, x, a + b x \right]$$

Program code:

```
Int[(c+d.*x_)^m_.*Gamma[n_,a+b_.*x_],x_Symbol] :=
  1/b*Subst[Int[(d*x/b)^m*Gamma[n,x],x],x,a+b*x] /;
FreeQ[{a,b,c,d,m,n},x] && EqQ[b*c-a*d,0]
```

$$2: \int \frac{\text{Gamma}[n, a + b x]}{c + d x} dx \text{ when } n - 1 \in \mathbb{Z}^+$$

Derivation: Algebraic expansion

$$\text{Basis: } \text{Gamma}[n, z] = \frac{z^{n-1}}{e^z} + (n - 1) \text{Gamma}[n - 1, z]$$

Rule: If  $n - 1 \in \mathbb{Z}^+$ , then

$$\int \frac{\text{Gamma}[n, a + b x]}{c + d x} dx \rightarrow \int \frac{(a + b x)^{n-1}}{(c + d x) e^{a+b x}} dx + (n - 1) \int \frac{\text{Gamma}[n - 1, a + b x]}{c + d x} dx$$

Program code:

```
Int[Gamma[n_,a_.+b_.*x_]/(c_.+d_.*x_),x_Symbol] :=
  Int[(a+b*x)^(n-1)/((c+d*x)*E^(a+b*x)),x] + (n-1)*Int[Gamma[n-1,a+b*x]/(c+d*x),x] /;
FreeQ[{a,b,c,d},x] && IGtQ[n,1]
```

$$\mathbf{3:} \int (c + d x)^m \text{Gamma}[n, a + b x] dx \text{ when } (m \in \mathbb{Z}^+ \vee n \in \mathbb{Z}^+ \vee (m | n) \in \mathbb{Z}) \wedge m \neq -1$$

Derivation: Integration by parts

$$\text{Basis: } \partial_x \text{Gamma}[n, a + b x] = - \frac{b (a + b x)^{n-1}}{e^{a + b x}}$$

Rule: If  $(m \in \mathbb{Z}^+ \vee n \in \mathbb{Z}^+ \vee (m | n) \in \mathbb{Z}) \wedge m \neq -1$ , then

$$\int (c + d x)^m \text{Gamma}[n, a + b x] dx \rightarrow \frac{(c + d x)^{m+1} \text{Gamma}[n, a + b x]}{d (m + 1)} + \frac{b}{d (m + 1)} \int \frac{(c + d x)^{m+1} (a + b x)^{n-1}}{e^{a + b x}} dx$$

Program code:

```
Int[(c_.+d_.*x_)^m_.*Gamma[n_,a_.+b_.*x_],x_Symbol] :=
Block[{$UseGamma=True},
(c+d*x)^(m+1)*Gamma[n,a+b*x]/(d*(m+1)) +
b/(d*(m+1))*Int[(c+d*x)^(m+1)*(a+b*x)^(n-1)/E^(a+b*x),x] /;
FreeQ[{a,b,c,d,m,n},x] && (IGtQ[m,0] || IGtQ[n,0] || IntegersQ[m,n]) && NeQ[m,-1]
```

$$\mathbf{U:} \int (c + d x)^m \text{Gamma}[n, a + b x] dx$$

Rule:

$$\int (c + d x)^m \text{Gamma}[n, a + b x] dx \rightarrow \int (c + d x)^m \text{Gamma}[n, a + b x] dx$$

Program code:

```
Int[(c_.+d_.*x_)^m_.*Gamma[n_,a_.+b_.*x_],x_Symbol] :=
Unintegrable[(c+d*x)^m*Gamma[n,a+b*x],x] /;
FreeQ[{a,b,c,d,m,n},x]
```

$$2. \int \text{LogGamma}[a + b x] \, dx$$

$$1: \int \text{LogGamma}[a + b x] \, dx$$

Derivation: Primitive rule

$$\text{Basis: } \frac{\partial \psi^{(-2)}(z)}{\partial z} = \log \Gamma(z)$$

Rule:

$$\int \text{LogGamma}[a + b x] \, dx \rightarrow \frac{\text{PolyGamma}[-2, a + b x]}{b}$$

Program code:

```
Int[LogGamma[a_.+b_.*x_],x_Symbol] :=
  PolyGamma[-2,a+b*x]/b /;
FreeQ[{a,b},x]
```



$$2. \int (c + d x)^m \text{LogGamma}[a + b x] dx$$

$$1: \int (c + d x)^m \text{LogGamma}[a + b x] dx \text{ when } m \in \mathbb{Z}^+$$

Derivation: Integration by parts

Rule: If  $m \in \mathbb{Z}^+$ , then

$$\int (c + d x)^m \text{LogGamma}[a + b x] dx \rightarrow \frac{(c + d x)^m \text{PolyGamma}[-2, a + b x]}{b} - \frac{d m}{b} \int (c + d x)^{m-1} \text{PolyGamma}[-2, a + b x] dx$$

Program code:

```
Int[(c_+d_*x_)^m_*LogGamma[a_+b_*x_],x_Symbol] :=
  (c+d*x)^m*PolyGamma[-2,a+b*x]/b -
  d*m/b*Int[(c+d*x)^(m-1)*PolyGamma[-2,a+b*x],x] /;
FreeQ[{a,b,c,d},x] && IGtQ[m,0]
```

$$U: \int (c + d x)^m \text{LogGamma}[a + b x] dx$$

Rule:

$$\int (c + d x)^m \text{LogGamma}[a + b x] dx \rightarrow \int (c + d x)^m \text{LogGamma}[a + b x] dx$$

Program code:

```
Int[(c_+d_*x_)^m_*LogGamma[a_+b_*x_],x_Symbol] :=
  Unintegrable[(c+d*x)^m*LogGamma[a+b*x],x] /;
FreeQ[{a,b,c,d,m},x]
```

$$3. \int u \text{PolyGamma}[n, a + b x] dx$$

$$1: \int \text{PolyGamma}[n, a + b x] dx$$

Derivation: Primitive rule

$$\text{Basis: } \frac{\partial \psi^{(n)}(z)}{\partial z} = \psi^{(n+1)}(z)$$

Rule:

$$\int \text{PolyGamma}[n, a + b x] dx \rightarrow \frac{\text{PolyGamma}[n - 1, a + b x]}{b}$$

Program code:

```
Int[PolyGamma[n_, a_ + b_.*x_], x_Symbol] :=
  PolyGamma[n-1, a+b*x]/b /;
  FreeQ[{a,b,n}, x]
```

$$2. \int (c + dx)^m \text{PolyGamma}[n, a + bx] dx$$

$$1: \int (c + dx)^m \text{PolyGamma}[n, a + bx] dx \text{ when } m > 0$$

Derivation: Integration by parts

Rule: If  $m > 0$ , then

$$\int (c + dx)^m \text{PolyGamma}[n, a + bx] dx \rightarrow \frac{(c + dx)^m \text{PolyGamma}[n - 1, a + bx]}{b} - \frac{d}{b} \int (c + dx)^{m-1} \text{PolyGamma}[n - 1, a + bx] dx$$

Program code:

```
Int[(c_+d_*x_)^m_*PolyGamma[n_,a_+b_*x_],x_Symbol] :=
  (c+d*x)^m*PolyGamma[n-1,a+b*x]/b - d*m/b*Int[(c+d*x)^(m-1)*PolyGamma[n-1,a+b*x],x] /;
FreeQ[{a,b,c,d,n},x] && GtQ[m,0]
```

$$2: \int (c + dx)^m \text{PolyGamma}[n, a + bx] dx \text{ when } m < -1$$

Derivation: Inverted integration by parts

Rule: If  $m < -1$ , then

$$\int (c + dx)^m \text{PolyGamma}[n, a + bx] dx \rightarrow \frac{(c + dx)^{m+1} \text{PolyGamma}[n, a + bx]}{d(m+1)} - \frac{b}{d(m+1)} \int (c + dx)^{m+1} \text{PolyGamma}[n+1, a + bx] dx$$

Program code:

```
Int[(c_+d_*x_)^m_*PolyGamma[n_,a_+b_*x_],x_Symbol] :=
  (c+d*x)^(m+1)*PolyGamma[n,a+b*x]/(d*(m+1)) -
  b/(d*(m+1))*Int[(c+d*x)^(m+1)*PolyGamma[n+1,a+b*x],x] /;
FreeQ[{a,b,c,d,n},x] && LtQ[m,-1]
```

$$\mathbf{U:} \int (c + d x)^m \text{PolyGamma}[n, a + b x] dx$$

Rule:

$$\int (c + d x)^m \text{PolyGamma}[n, a + b x] dx \rightarrow \int (c + d x)^m \text{PolyGamma}[n, a + b x] dx$$

Program code:

```
Int[(c_+d_*x_)^m_*PolyGamma[n_,a_+b_*x_],x_Symbol] :=
  Unintegrable[(c+d*x)^m*PolyGamma[n,a+b*x],x] /;
  FreeQ[{a,b,c,d,m,n},x]
```

$$\mathbf{4:} \int \text{Gamma}[a + b x]^n \text{PolyGamma}[\theta, a + b x] dx$$

Derivation: Primitive rule

$$\text{Basis: } \frac{\partial \Gamma(z)^n}{\partial z} = n \psi^{(0)}(z) \Gamma(z)^n$$

Rule:

$$\int \text{Gamma}[a + b x]^n \text{PolyGamma}[\theta, a + b x] dx \rightarrow \frac{\text{Gamma}[a + b x]^n}{b n}$$

Program code:

```
Int[Gamma[a_+b_*x_]^n_*PolyGamma[theta,a_+b_*x_],x_Symbol] :=
  Gamma[a+b*x]^n/(b*n) /;
  FreeQ[{a,b,n},x]
```

5:  $\int ((a + b x)!)^n \text{PolyGamma}[\theta, c + b x] dx$  when  $c == a + 1$

Derivation: Primitive rule

Basis:  $\frac{\partial(z!)^n}{\partial z} = n \psi^{(0)}(z+1) (z!)^n$

Rule: If  $c == a + 1$ , then

$$\int ((a + b x)!)^n \text{PolyGamma}[\theta, c + b x] dx \rightarrow \frac{((a + b x)!)^n}{b n}$$

Program code:

```
Int[((a_+b_*x_)!)^n_*PolyGamma[θ,c_+b_*x_],x_Symbol] :=
  ((a+b*x)!)^n/(b*n) /;
  FreeQ[{a,b,c,n},x] && EqQ[c,a+1]
```

$$6. \int u \text{Gamma}[p, d(a + b \text{Log}[c x^n])] dx$$

$$1: \int \text{Gamma}[p, d(a + b \text{Log}[c x^n])] dx$$

Derivation: Integration by parts

$$\text{Basis: } \partial_x \text{Gamma}[p, d(a + b \text{Log}[c x^n])] == - \frac{b d n e^{-a} (d(a + b \text{Log}[c x^n]))^{p-1}}{x (c x^n)^{b d}}$$

Rule:

$$\int \text{Gamma}[p, d(a + b \text{Log}[c x^n])] dx \rightarrow x \text{Gamma}[p, d(a + b \text{Log}[c x^n])] + b d n e^{-a d} \int \frac{(d(a + b \text{Log}[c x^n]))^{p-1}}{(c x^n)^{b d}} dx$$

Program code:

```
Int[Gamma[p_,d_.*(a_.+b_.*Log[c_.*x_^n_.]),x_Symbol] :=
  x*Gamma[p,d*(a+b*Log[c*x^n])] + b*d*n*E^(-a*d)*Int[(d*(a+b*Log[c*x^n]))^(p-1)/(c*x^n)^(b*d),x] /;
FreeQ[{a,b,c,d,n,p},x]
```

$$2: \int \frac{\text{Gamma}[p, d (a + b \text{Log}[c x^n])]}{x} dx$$

Derivation: Integration by substitution

$$\text{Basis: } \frac{F[\text{Log}[c x^n]]}{x} == \frac{1}{n} \text{Subst}[F[x], x, \text{Log}[c x^n]] \partial_x \text{Log}[c x^n]$$

Rule:

$$\int \frac{\text{Gamma}[p, d (a + b \text{Log}[c x^n])]}{x} dx \rightarrow \frac{1}{n} \text{Subst}[\text{Gamma}[p, d (a + b x)], x, \text{Log}[c x^n]]$$

Program code:

```
Int[Gamma[p_, d_. * (a_. + b_. * Log[c_. * x_^n_.])] / x_, x_Symbol] :=
  1/n * Subst[Gamma[p, d * (a + b * x)], x, Log[c * x^n]] /;
FreeQ[{a, b, c, d, n, p}, x]
```

$$3: \int (e x)^m \text{Gamma}[p, d (a + b \text{Log}[c x^n])] dx \text{ when } m \neq -1$$

Derivation: Integration by parts

$$\text{Basis: } \partial_x \text{Gamma}[p, d (a + b \text{Log}[c x^n])] == - \frac{b d n e^{-a d} (d (a + b \text{Log}[c x^n]))^{-1+p}}{x (c x^n)^{b d}}$$

Rule: If  $m \neq -1$ , then

$$\int (e x)^m \text{Gamma}[p, d (a + b \text{Log}[c x^n])] dx \rightarrow \frac{(e x)^{m+1} \text{Gamma}[p, d (a + b \text{Log}[c x^n])]}{e (m+1)} + \frac{b d n e^{-a d} (e x)^{b d n}}{(m+1) (c x^n)^{b d}} \int (e x)^{m-b d n} (d (a + b \text{Log}[c x^n]))^{p-1} dx$$

Program code:

```
Int[(e.*x_)^m_.*Gamma[p_,d_.*(a_+b_.*Log[c_.*x_^n_.])],x_Symbol] :=
(e*x)^(m+1)*Gamma[p,d*(a+b*Log[c*x^n])]/(e*(m+1)) +
b*d*n*E^(-a*d)*(e*x)^(b*d*n)/((m+1)*(c*x^n)^(b*d))*Int[(e*x)^(m-b*d*n)*(d*(a+b*Log[c*x^n]))^(p-1),x] /;
FreeQ[{a,b,c,d,e,m,n,p},x] && NeQ[m,-1]
```



$$7. \int u \text{Gamma}[p, f(a + b \text{Log}[c(d + ex)^n])] dx$$

$$1: \int \text{Gamma}[p, f(a + b \text{Log}[c(d + ex)^n])] dx$$

Derivation: Integration by substitution

Rule:

$$\int \text{Gamma}[p, f(a + b \text{Log}[c(d + ex)^n])] dx \rightarrow \frac{1}{e} \text{Subst}\left[\int \text{Gamma}[p, f(a + b \text{Log}[cx^n])] dx, x, d + ex\right]$$

Program code:

```
Int[Gamma[p_, f_.*(a_.+b_.*Log[c_.*(d+e.*x_)^n_.)], x_Symbol] :=
  1/e*Subst[Int[Gamma[p, f*(a+b*Log[c*x^n])], x], x, d+e*x] /;
  FreeQ[{a, b, c, d, e, f, n, p}, x]
```

$$2: \int (g + hx)^m \text{Gamma}[p, f(a + b \text{Log}[c(d + ex)^n])] dx \text{ when } eg - dh = 0$$

Derivation: Integration by substitution

$$\text{Basis: If } eg - dh = 0, \text{ then } (g + hx)^m F[d + ex] = \frac{1}{e} \text{Subst}\left[\left(\frac{gx}{d}\right)^m F[x], x, d + ex\right] \partial_x (d + ex)$$

Rule: If  $eg - dh = 0$ , then

$$\int (g + hx)^m \text{Gamma}[p, f(a + b \text{Log}[c(d + ex)^n])] dx \rightarrow \frac{1}{e} \text{Subst}\left[\int \left(\frac{gx}{d}\right)^m \text{Gamma}[p, f(a + b \text{Log}[cx^n])] dx, x, d + ex\right]$$

Program code:

```
Int[(g+h.*x_)^m_.*Gamma[p_, f_.*(a_.+b_.*Log[c_.*(d+e.*x_)^n_.)], x_Symbol] :=
  1/e*Subst[Int[(g*x/d)^m*Gamma[p, f*(a+b*Log[c*x^n])], x], x, d+e*x] /;
  FreeQ[{a, b, c, d, e, f, g, h, m, n, p}, x] && EqQ[e*g-d*h, 0]
```

