

## Rules for integrands of the form $(f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n$

1.  $\int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$  when  $e = c^2 d$

1.  $\int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$  when  $e = c^2 d \wedge n > 0$

1.  $\int x (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$  when  $e = c^2 d \wedge n > 0$

1:  $\int \frac{x (a + b \operatorname{ArcSinh}[c x])^n}{d + e x^2} dx$  when  $e = c^2 d \wedge n \in \mathbb{Z}^+$

Derivation: Integration by substitution

Basis: If  $e = c^2 d$ , then  $\frac{x}{d+ex^2} = \frac{1}{e} \operatorname{Subst}[\operatorname{Tanh}[x], x, \operatorname{ArcSinh}[c x]] \partial_x \operatorname{ArcSinh}[c x]$

Basis: If  $c^2 d + e = 0$ , then  $\frac{x}{d+ex^2} = \frac{1}{e} \operatorname{Subst}[\operatorname{Coth}[x], x, \operatorname{ArcCosh}[c x]] \partial_x \operatorname{ArcCosh}[c x]$

Note: If  $n \in \mathbb{Z}^+$ , then  $(a + b x)^n \operatorname{Tanh}[x]$  is integrable in closed-form.

Rule: If  $e = c^2 d \wedge n \in \mathbb{Z}^+$ , then

$$\int \frac{x (a + b \operatorname{ArcSinh}[c x])^n}{d + e x^2} dx \rightarrow \frac{1}{e} \operatorname{Subst}\left[\int (a + b x)^n \operatorname{Tanh}[x] dx, x, \operatorname{ArcSinh}[c x]\right]$$

Program code:

```
Int[x_*(a_.*b_.*ArcSinh[c_.*x_])^n_./(d_+e_.*x_^2),x_Symbol] :=
  1/e*Subst[Int[(a+b*x)^n*Tanh[x],x],x,ArcSinh[c*x]] /;
FreeQ[{a,b,c,d,e},x] && EqQ[e,c^2*d] && IGtQ[n,0]
```

```
Int[x_*(a_.*b_.*ArcCosh[c_.*x_])^n_./(d_+e_.*x_^2),x_Symbol] :=
  1/e*Subst[Int[(a+b*x)^n*Coth[x],x],x,ArcCosh[c*x]] /;
FreeQ[{a,b,c,d,e},x] && EqQ[c^2*d+e,0] && IGtQ[n,0]
```

$$2. \int x (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p \neq -1$$

$$1: \int x (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p \neq -1 \wedge (p \in \mathbb{Z} \vee d > 0)$$

Derivation: Integration by parts

Rule: If  $e = c^2 d \wedge n > 0 \wedge p \neq -1 \wedge (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int x (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(d + e x^2)^{p+1} (a + b \operatorname{ArcSinh}[c x])^n}{2 e (p + 1)} - \frac{b n d^p}{2 c (p + 1)} \int (1 + c^2 x^2)^{p+\frac{1}{2}} (a + b \operatorname{ArcSinh}[c x])^{n-1} dx$$

Program code:

```
(* Int[x*(d+e*x^2)^p.*(a+b.*ArcSinh[c.*x])^n.,x_Symbol] :=
  (d+e*x^2)^(p+1)*(a+b.*ArcSinh[c*x])^n/(2*e*(p+1)) -
  b*n*d^p/(2*c*(p+1))*Int[(1+c^2*x^2)^(p+1/2)*(a+b.*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,p},x] && EqQ[e,c^2*d] && GtQ[n,0] && NeQ[p,-1] && (IntegerQ[p] || GtQ[d,0]) *)
```

```
Int[x*(d+e*x^2)^p.*(a+b.*ArcCosh[c.*x])^n.,x_Symbol] :=
  (d+e*x^2)^(p+1)*(a+b.*ArcCosh[c*x])^n/(2*e*(p+1)) -
  b*n*(-d)^p/(2*c*(p+1))*Int[(1+c*x)^(p+1/2)*(-1+c*x)^(p+1/2)*(a+b.*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,p},x] && EqQ[c^2*d+e,0] && GtQ[n,0] && NeQ[p,-1] && IntegerQ[p]
```

```
(* Int[x*(d1+e1*x)^p.*(d2+e2*x)^p.*(a+b.*ArcCosh[c.*x])^n.,x_Symbol] :=
  (d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b.*ArcCosh[c*x])^n/(2*e1*e2*(p+1)) -
  b*n*(-d1*d2)^(p-1/2)/(2*c*(p+1))*Int[(-1+c^2*x^2)^(p+1/2)*(a+b.*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && IntegerQ[p+1/2] && (GtQ[d1,0] && LtQ[d2,0]) *)
```

$$2: \int x (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p \neq -1$$

Derivation: Integration by parts and piecewise constant extraction

Basis: If  $e = c^2 d$ , then  $\partial_x \frac{(d+e x^2)^p}{(1+c^2 x^2)^p} = 0$

Rule: If  $e = c^2 d \wedge n > 0 \wedge p \neq -1$ , then

$$\int x (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])^n}{2 e (p+1)} - \frac{b n d^{\operatorname{IntPart}[p]} (d+e x^2)^{\operatorname{FracPart}[p]}}{2 c (p+1) (1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

Program code:

```
Int[x_*(d_+e_.*x_^2)^p_.*(a_.+b_.*ArcSinh[c_.*x_])^n_.,x_Symbol] :=
  (d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n/(2*e*(p+1)) -
  b*n*d^IntPart[p]*(d+e*x^2)^FracPart[p]/(2*c*(p+1)*(1+c^2*x^2)^FracPart[p])*
  Int[(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,p},x] && EqQ[e,c^2*d] && GtQ[n,0] && NeQ[p,-1]
```

```
Int[x_*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_.*(a_.+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  (d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n/(2*e1*e2*(p+1)) -
  b*n*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(2*c*(p+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[(-1+c^2*x^2)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && NeQ[p,-1] && IntegerQ[p+1/2]
```

```
Int[x_*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_.*(a_.+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  (d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n/(2*e1*e2*(p+1)) -
  b*n*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(2*c*(p+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[(1+c*x)^(p+1/2)*(-1+c*x)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && NeQ[p,-1]
```

2.  $\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx$  when  $e = c^2 d \wedge n > 0 \wedge m + 2p + 3 = 0$

1:  $\int \frac{(a+b \operatorname{ArcSinh}[c x])^n}{x (d+e x^2)} dx$  when  $e = c^2 d \wedge n \in \mathbb{Z}^+$

- Derivation: Integration by substitution

Basis: If  $e = c^2 d$ , then  $\frac{1}{x (d+e x^2)} = \frac{1}{d} \operatorname{Subst}\left[\frac{1}{\operatorname{Cosh}[x] \operatorname{Sinh}[x]}, x, \operatorname{ArcSinh}[c x]\right] \partial_x \operatorname{ArcSinh}[c x]$

- Rule: If  $e = c^2 d \wedge n \in \mathbb{Z}^+$ , then

$$\int \frac{(a+b \operatorname{ArcSinh}[c x])^n}{x (d+e x^2)} dx \rightarrow \frac{1}{d} \operatorname{Subst}\left[\int \frac{(a+b x)^n}{\operatorname{Cosh}[x] \operatorname{Sinh}[x]} dx, x, \operatorname{ArcSinh}[c x]\right]$$

- Program code:

```
Int[(a_.+b_.*ArcSinh[c_.*x_])^n_./ (x_*(d_+e_.*x_^2)), x_Symbol] :=
  1/d*Subst[Int[(a+b*x)^n/(Cosh[x]*Sinh[x]), x], x, ArcSinh[c*x]] /;
FreeQ[{a,b,c,d,e}, x] && EqQ[e, c^2*d] && IGtQ[n, 0]
```

```
Int[(a_.+b_.*ArcCosh[c_.*x_])^n_./ (x_*(d_+e_.*x_^2)), x_Symbol] :=
  -1/d*Subst[Int[(a+b*x)^n/(Cosh[x]*Sinh[x]), x], x, ArcCosh[c*x]] /;
FreeQ[{a,b,c,d,e}, x] && EqQ[c^2*d+e, 0] && IGtQ[n, 0]
```

$$2. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge m+2p+3 = 0 \wedge m \neq -1$$

$$1: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge m+2p+3 = 0 \wedge m \neq -1 \wedge (p \in \mathbb{Z} \vee d > 0)$$

## Derivation: Integration by parts

Basis: If  $m+2p+3 = 0$ , then  $(f x)^m (d+e x^2)^p = \partial_x \frac{(f x)^{m+1} (d+e x^2)^{p+1}}{d f (m+1)}$

Rule: If  $e = c^2 d \wedge n > 0 \wedge m+2p+3 = 0 \wedge m \neq -1 \wedge (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(f x)^{m+1} (d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])^n}{d f (m+1)} - \frac{b c n d^p}{f (m+1)} \int (f x)^{m+1} (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

## Program code:

```
(* Int[(f_.x_)^m_*(d+e_.x_^2)^p_*(a_.+b_.*ArcSinh[c_.x_])^n_.,x_Symbol] :=
  (f*x)^(m+1)*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n/(d*f*(m+1)) -
  b*c*n*d^p/(f*(m+1))*Int[(f*x)^(m+1)*(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,m,p},x] && EqQ[e,c^2*d] && GtQ[n,0] && EqQ[m+2*p+3,0] && NeQ[m,-1] && (IntegerQ[p] || GtQ[d,0]) *)
```

```
Int[(f_.x_)^m_*(d+e_.x_^2)^p_*(a_.+b_.*ArcCosh[c_.x_])^n_.,x_Symbol] :=
  (f*x)^(m+1)*(d+e*x^2)^(p+1)*(a+b*ArcCosh[c*x])^n/(d*f*(m+1)) +
  b*c*n*(-d)^p/(f*(m+1))*Int[(f*x)^(m+1)*(1+c*x)^(p+1/2)*(-1+c*x)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,m,p},x] && EqQ[c^2*d+e,0] && GtQ[n,0] && EqQ[m+2*p+3,0] && NeQ[m,-1] && IntegerQ[p]
```

```
(* Int[(f_.x_)^m_*(d1+e1_.x_)^p_*(d2+e2_.x_)^p_*(a_.+b_.*ArcCosh[c_.x_])^n_.,x_Symbol] :=
  (f*x)^(m+1)*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n/(d1*d2*f*(m+1)) +
  b*c*n*(-d1*d2)^p/(f*(m+1))*Int[(f*x)^(m+1)*(-1+c^2*x^2)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && EqQ[m+2*p+3,0] &&
  NeQ[m,-1] && IntegerQ[p+1/2] && (GtQ[d1,0] && LtQ[d2,0]) *)
```

$$2: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge m+2p+3 = 0 \wedge m \neq -1$$

Derivation: Integration by parts and piecewise constant extraction

$$\text{Basis: If } m+2p+3 = 0, \text{ then } (f x)^m (d+e x^2)^p = \partial_x \frac{(f x)^{m+1} (d+e x^2)^{p+1}}{d f (m+1)}$$

$$\text{Basis: If } e = c^2 d, \text{ then } \partial_x \frac{(d+e x^2)^p}{(1+c^2 x^2)^p} = 0$$

Rule: If  $e = c^2 d \wedge n > 0 \wedge m+2p+3 = 0 \wedge m \neq -1$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(f x)^{m+1} (d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])^n}{d f (m+1)} - \frac{b c n d^{\operatorname{IntPart}[p]} (d+e x^2)^{\operatorname{FracPart}[p]}}{f (m+1) (1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int (f x)^{m+1} (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

Program code:

```
Int[(f_.*x_)^m_*(d_+e_.*x_^2)^p_*(a_+b_.*ArcSinh[c_.*x_])^n_.,x_Symbol] :=
  (f*x)^(m+1)*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n/(d*f*(m+1)) -
  b*c*n*d^IntPart[p]*(d+e*x^2)^FracPart[p]/(f*(m+1)*(1+c^2*x^2)^FracPart[p])*
  Int[(f*x)^(m+1)*(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,m,p},x] && EqQ[e,c^2*d] && GtQ[n,0] && EqQ[m+2*p+3,0] && NeQ[m,-1]
```

```
Int[(f_.*x_)^m_*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  (f*x)^(m+1)*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n/(d1*d2*f*(m+1)) +
  b*c*n*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(f*(m+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[(f*x)^(m+1)*(-1+c^2*x^2)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && EqQ[m+2*p+3,0] && NeQ[m,-1] && IntegerQ[p+1/2]
```

```
Int[(f_.*x_)^m_*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  (f*x)^(m+1)*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n/(d1*d2*f*(m+1)) +
  b*c*n*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(f*(m+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[(f*x)^(m+1)*(1+c*x)^(p+1/2)*(-1+c*x)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && EqQ[m+2*p+3,0] && NeQ[m,-1]
```

$$3. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \text{ when } e = c^2 d \wedge p \in \mathbb{Z}^+$$

$$1. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \text{ when } e = c^2 d \wedge p \in \mathbb{Z}^+ \wedge \frac{m-1}{2} \in \mathbb{Z}^-$$

$$1: \int \frac{(d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])}{x} dx \text{ when } e = c^2 d \wedge p \in \mathbb{Z}^+$$

## Derivation: Inverted integration by parts

Rule: If  $e = c^2 d \wedge p \in \mathbb{Z}^+$ , then

$$\int \frac{(d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])}{x} dx \rightarrow \frac{(d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])}{2p} - \frac{b c d^p}{2p} \int (1+c^2 x^2)^{p-\frac{1}{2}} dx + d \int \frac{(d+e x^2)^{p-1} (a+b \operatorname{ArcSinh}[c x])}{x} dx$$

## Program code:

```
Int[(d_+e_.*x_^2)^p_.*(a_.+b_.*ArcSinh[c_.*x_])/x_,x_Symbol] :=
  (d+e*x^2)^p*(a+b*ArcSinh[c*x])/(2*p) -
  b*c*d^p/(2*p)*Int[(1+c^2*x^2)^(p-1/2),x] +
  d*Int[(d+e*x^2)^(p-1)*(a+b*ArcSinh[c*x])/x,x] /;
FreeQ[{a,b,c,d,e},x] && EqQ[e,c^2*d] && IGtQ[p,0]
```

```
Int[(d_+e_.*x_^2)^p_.*(a_.+b_.*ArcCosh[c_.*x_])/x_,x_Symbol] :=
  (d+e*x^2)^p*(a+b*ArcCosh[c*x])/(2*p) -
  b*c*(-d)^p/(2*p)*Int[(1+c*x)^(p-1/2)*(-1+c*x)^(p-1/2),x] +
  d*Int[(d+e*x^2)^(p-1)*(a+b*ArcCosh[c*x])/x,x] /;
FreeQ[{a,b,c,d,e},x] && EqQ[c^2*d+e,0] && IGtQ[p,0]
```

$$2: \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x]) dx \text{ when } e = c^2 d \wedge p \in \mathbb{Z}^+ \wedge \frac{m+1}{2} \in \mathbb{Z}^-$$

Derivation: Inverted integration by parts

Rule: If  $e = c^2 d \wedge p \in \mathbb{Z}^+ \wedge \frac{m+1}{2} \in \mathbb{Z}^-$ , then

$$\int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x]) dx \rightarrow \frac{(f x)^{m+1} (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])}{f (m+1)} - \frac{b c d^p}{f (m+1)} \int (f x)^{m+1} (1 + c^2 x^2)^{p-\frac{1}{2}} dx - \frac{2 e p}{f^2 (m+1)} \int (f x)^{m+2} (d + e x^2)^{p-1} (a + b \operatorname{ArcSinh}[c x]) dx$$

Program code:

```
Int[(f_.**x_)^m_*(d_+e_.*x_^2)^p_.*(a_.+b_.*ArcSinh[c_.*x_]),x_Symbol] :=
  (f**x)^(m+1)*(d+e*x^2)^p*(a+b*ArcSinh[c*x])/(f*(m+1)) -
  b*c*d^p/(f*(m+1))*Int[(f**x)^(m+1)*(1+c^2*x^2)^(p-1/2),x] -
  2*e*p/(f^2*(m+1))*Int[(f**x)^(m+2)*(d+e*x^2)^(p-1)*(a+b*ArcSinh[c*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[e,c^2*d] && IGtQ[p,0] && ILtQ[(m+1)/2,0]
```

```
Int[(f_.**x_)^m_*(d_+e_.*x_^2)^p_.*(a_.+b_.*ArcCosh[c_.*x_]),x_Symbol] :=
  (f**x)^(m+1)*(d+e*x^2)^p*(a+b*ArcCosh[c*x])/(f*(m+1)) -
  b*c*(-d)^p/(f*(m+1))*Int[(f**x)^(m+1)*(1+c*x)^(p-1/2)*(-1+c*x)^(p-1/2),x] -
  2*e*p/(f^2*(m+1))*Int[(f**x)^(m+2)*(d+e*x^2)^(p-1)*(a+b*ArcCosh[c*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[c^2*d+e,0] && IGtQ[p,0] && ILtQ[(m+1)/2,0]
```



$$2: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \text{ when } e = c^2 d \wedge p \in \mathbb{Z}^+$$

Derivation: Integration by parts

Rule: If  $e = c^2 d \wedge p \in \mathbb{Z}^+$ , let  $u = \int (f x)^m (d+e x^2)^p dx$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \rightarrow u (a+b \operatorname{ArcSinh}[c x]) - b c \int \frac{u}{\sqrt{1+c^2 x^2}} dx$$

Program code:

```
Int[(f_.**x_)^m_*(d_+e_.**x_^2)^p_.*(a_+b_.**ArcSinh[c_.**x_]),x_Symbol] :=
  With[{u=IntHide[(f**x)^m*(d+e*x^2)^p,x]},
    Dist[a+b*ArcSinh[c*x],u,x] - b*c*Int[SimplifyIntegrand[u/Sqrt[1+c^2*x^2],x],x] /;
  FreeQ[{a,b,c,d,e,f,m},x] && EqQ[e,c^2*d] && IGtQ[p,0]
```

```
Int[(f_.**x_)^m_*(d_+e_.**x_^2)^p_.*(a_+b_.**ArcCosh[c_.**x_]),x_Symbol] :=
  With[{u=IntHide[(f**x)^m*(d+e*x^2)^p,x]},
    Dist[a+b*ArcCosh[c*x],u,x] - b*c*Int[SimplifyIntegrand[u/(Sqrt[1+c*x]*Sqrt[-1+c*x]),x],x] /;
  FreeQ[{a,b,c,d,e,f,m},x] && EqQ[c^2*d+e,0] && IGtQ[p,0]
```

$$4. \int x^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \text{ when } e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z} \wedge \left( \frac{m+1}{2} \in \mathbb{Z}^+ \vee \frac{m+2p+3}{2} \in \mathbb{Z}^- \right)$$

$$1: \int x^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \text{ when } e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z} \wedge \left( \frac{m+1}{2} \in \mathbb{Z}^+ \vee \frac{m+2p+3}{2} \in \mathbb{Z}^- \right) \wedge p \neq -\frac{1}{2} \wedge d > 0$$

Derivation: Integration by parts

Note: If  $p - \frac{1}{2} \in \mathbb{Z} \wedge \left( \frac{m+1}{2} \in \mathbb{Z}^+ \vee \frac{m+2p+3}{2} \in \mathbb{Z}^- \right)$ , then  $\int x^m (1+c^2 x^2)^p dx$  is an algebraic function not involving logarithms, inverse trig or inverse hyperbolic functions.

Rule: If  $e = c^2 d \wedge p - \frac{1}{2} \in \mathbb{Z} \wedge \left( \frac{m+1}{2} \in \mathbb{Z}^+ \vee \frac{m+2p+3}{2} \in \mathbb{Z}^- \right) \wedge p \neq -\frac{1}{2} \wedge d > 0$ , let  $u = \int x^m (1+c^2 x^2)^p dx$ , then

$$\int x^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \rightarrow d^p u (a+b \operatorname{ArcSinh}[c x]) - b c d^p \int \frac{u}{\sqrt{1+c^2 x^2}} dx$$

Program code:

```
Int[x^m*(d+e.*x^2)^p*(a.+b.*ArcSinh[c.*x]),x_Symbol] :=
  With[{u=IntHide[x^m*(1+c^2*x^2)^p,x]},
    Dist[d^p*(a+b*ArcSinh[c*x]),u,x] - b*c*d^p*Int[SimplifyIntegrand[u/Sqrt[1+c^2*x^2],x],x] /;
    FreeQ[{a,b,c,d,e},x] && EqQ[e,c^2*d] && IntegerQ[p-1/2] && (IGtQ[(m+1)/2,0] || ILtQ[(m+2*p+3)/2,0]) && NeQ[p,-1/2] && GtQ[d,0]
```

```
Int[x^m*(d1+e1.*x)^p*(d2+e2.*x)^p*(a.+b.*ArcCosh[c.*x]),x_Symbol] :=
  With[{u=IntHide[x^m*(1+c*x)^p*(-1+c*x)^p,x]},
    Dist[(-d1*d2)^p*(a+b*ArcCosh[c*x]),u,x] - b*c*(-d1*d2)^p*Int[SimplifyIntegrand[u/(Sqrt[1+c*x]*Sqrt[-1+c*x]),x],x] /;
    FreeQ[{a,b,c,d1,e1,d2,e2},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IntegerQ[p-1/2] &&
    (IGtQ[(m+1)/2,0] || ILtQ[(m+2*p+3)/2,0]) && NeQ[p,-1/2] && GtQ[d1,0] && LtQ[d2,0]
```

$$2: \int x^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \text{ when } e = c^2 d \wedge p + \frac{1}{2} \in \mathbb{Z}^+ \wedge \left( \frac{m+1}{2} \in \mathbb{Z}^+ \vee \frac{m+2p+3}{2} \in \mathbb{Z}^- \right)$$

Derivation: Integration by parts and piecewise constant extraction

Note: If  $p + \frac{1}{2} \in \mathbb{Z} \wedge \left( \frac{m+1}{2} \in \mathbb{Z}^+ \vee \frac{m+2p+3}{2} \in \mathbb{Z}^- \right)$ , then  $\int x^m (1+c^2 x^2)^p dx$  is an algebraic function not involving logarithms, inverse trig or inverse hyperbolic functions.

Rule: If  $e = c^2 d \wedge p + \frac{1}{2} \in \mathbb{Z}^+ \wedge \left( \frac{m+1}{2} \in \mathbb{Z}^+ \vee \frac{m+2p+3}{2} \in \mathbb{Z}^- \right)$ , let  $u = \int x^m (1+c^2 x^2)^p dx$ , then

$$\int x^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \rightarrow (a+b \operatorname{ArcSinh}[c x]) \int x^m (d+e x^2)^p dx - \frac{b c d^{p-\frac{1}{2}} \sqrt{d+e x^2}}{\sqrt{1+c^2 x^2}} \int \frac{u}{\sqrt{1+c^2 x^2}} dx$$

Program code:

```
Int[x^m*(d+e.*x^2)^p*(a.+b.*ArcSinh[c.*x]),x_Symbol] :=
  With[{u=IntHide[x^m*(1+c^2*x^2)^p,x]},
    (a+b*ArcSinh[c*x])*Int[x^m*(d+e*x^2)^p,x] -
    b*c*d^(p-1/2)*Sqrt[d+e*x^2]/Sqrt[1+c^2*x^2]*Int[SimplifyIntegrand[u/Sqrt[1+c^2*x^2],x],x] /;
  FreeQ[{a,b,c,d,e},x] && EqQ[e,c^2*d] && IGtQ[p+1/2,0] && (IGtQ[(m+1)/2,0] || ILtQ[(m+2*p+3)/2,0])
```

```
Int[x^m*(d1+e1.*x)^p*(d2+e2.*x)^p*(a.+b.*ArcCosh[c.*x]),x_Symbol] :=
  With[{u=IntHide[x^m*(1+c*x)^p*(-1+c*x)^p,x]},
    (a+b*ArcCosh[c*x])*Int[x^m*(d1+e1*x)^p*(d2+e2*x)^p,x] -
    b*c*(-d1*d2)^(p-1/2)*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]/(Sqrt[1+c*x]*Sqrt[-1+c*x])*
    Int[SimplifyIntegrand[u/(Sqrt[1+c*x]*Sqrt[-1+c*x]),x],x] /;
  FreeQ[{a,b,c,d1,e1,d2,e2},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IGtQ[p+1/2,0] && (IGtQ[(m+1)/2,0] || ILtQ[(m+2*p+3)/2,0])
```

$$5. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p > 0$$

$$1. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m < -1$$

$$1: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m < -1 \wedge (p \in \mathbb{Z} \vee d > 0)$$

## Derivation: Inverted integration by parts

Rule: If  $e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m < -1 \wedge (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(f x)^{m+1} (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n}{f (m+1)} - \frac{2 e p}{f^2 (m+1)} \int (f x)^{m+2} (d+e x^2)^{p-1} (a+b \operatorname{ArcSinh}[c x])^n dx - \frac{b c n d^p}{f (m+1)} \int (f x)^{m+1} (1+c^2 x^2)^{p-\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

## Program code:

```
(* Int[(f_*x_)^m_*(d_+e_*x_^2)^p_*(a_+b_*ArcSinh[c_*x_])^n_,x_Symbol] :=
(f*x)^(m+1)*(d+e*x^2)^p*(a+b*ArcSinh[c*x])^n/(f*(m+1)) -
2*e*p/(f^2*(m+1))*Int[(f*x)^(m+2)*(d+e*x^2)^(p-1)*(a+b*ArcSinh[c*x])^n,x] -
b*c*n*d^p/(f*(m+1))*Int[(f*x)^(m+1)*(1+c^2*x^2)^(p-1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[e,c^2*d] && GtQ[n,0] && GtQ[p,0] && LtQ[m,-1] && (IntegerQ[p] || GtQ[d,0]) *)
```

```
Int[(f_*x_)^m_*(d_+e_*x_^2)^p_*(a_+b_*ArcCosh[c_*x_])^n_,x_Symbol] :=
(f*x)^(m+1)*(d+e*x^2)^p*(a+b*ArcCosh[c*x])^n/(f*(m+1)) -
2*e*p/(f^2*(m+1))*Int[(f*x)^(m+2)*(d+e*x^2)^(p-1)*(a+b*ArcCosh[c*x])^n,x] -
b*c*n*(-d)^p/(f*(m+1))*Int[(f*x)^(m+1)*(1+c*x)^(p-1/2)*(-1+c*x)^(p-1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[c^2*d+e,0] && GtQ[n,0] && GtQ[p,0] && LtQ[m,-1] && IntegerQ[p]
```

```
(* Int [(f_.*x_)^m_*(d1_+e1_.*x_)^p_*(d2_+e2_.*x_)^q_*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
(f*x)^(m+1)*(d1+e1*x)^p*(d2+e2*x)^q*(a+b*ArcCosh[c*x])^n/(f*(m+1)) -
2*e1*e2*p/(f^2*(m+1))*Int[(f*x)^(m+2)*(d1+e1*x)^(p-1)*(d2+e2*x)^(q-1)*(a+b*ArcCosh[c*x])^n,x] -
b*c*n*(-d1*d2)^p/(f*(m+1))*Int[(f*x)^(m+1)*(-1+c^2*x^2)^(p-1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && GtQ[p,0] && LtQ[m,-1] &&
IntegerQ[p-1/2] && (GtQ[d1,0] && LtQ[d2,0]) *)
```

$$2. \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m < -1$$

$$1: \int (f x)^m \sqrt{d + e x^2} (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge m < -1$$

Derivation: Inverted integration by parts

Note: The piecewise constant factor in the second integral reduces the degree of  $d$  in the resulting antiderivative.

Rule: If  $e = c^2 d \wedge n > 0 \wedge m < -1$ , then

$$\int (f x)^m \sqrt{d + e x^2} (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(f x)^{m+1} \sqrt{d + e x^2} (a + b \operatorname{ArcSinh}[c x])^n}{f (m+1)} - \frac{b c n \sqrt{d + e x^2}}{f (m+1) \sqrt{1 + c^2 x^2}} \int (f x)^{m+1} (a + b \operatorname{ArcSinh}[c x])^{n-1} dx - \frac{c^2 \sqrt{d + e x^2}}{f^2 (m+1) \sqrt{1 + c^2 x^2}} \int \frac{(f x)^{m+2} (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{1 + c^2 x^2}} dx$$

Program code:

```
Int [(f_.*x_)^m_*Sqrt[d_+e_.*x_^2]*(a_+b_.*ArcSinh[c_.*x_])^n_.,x_Symbol] :=
(f*x)^(m+1)*Sqrt[d+e*x^2]*(a+b*ArcSinh[c*x])^n/(f*(m+1)) -
b*c*n*Sqrt[d+e*x^2]/(f*(m+1)*Sqrt[1+c^2*x^2])*Int[(f*x)^(m+1)*(a+b*ArcSinh[c*x])^(n-1),x] -
c^2*Sqrt[d+e*x^2]/(f^2*(m+1)*Sqrt[1+c^2*x^2])*Int[(f*x)^(m+2)*(a+b*ArcSinh[c*x])^n/Sqrt[1+c^2*x^2],x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[e,c^2*d] && GtQ[n,0] && LtQ[m,-1]
```

```

Int [(f_*x_)^m_*Sqrt[d1_+e1_*x_]_*Sqrt[d2_+e2_*x_] *(a_+b_*ArcCosh[c_*x_])^n_.,x_Symbol] :=
(f*x)^(m+1)*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]*(a+b*ArcCosh[c*x])^n/(f*(m+1)) -
b*c*n*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]/(f*(m+1)*Sqrt[1+c*x]*Sqrt[-1+c*x])*
Int[(f*x)^(m+1)*(a+b*ArcCosh[c*x])^(n-1),x] -
c^2*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]/(f^2*(m+1)*Sqrt[1+c*x]*Sqrt[-1+c*x])*
Int[((f*x)^(m+2)*(a+b*ArcCosh[c*x])^n)/(Sqrt[1+c*x]*Sqrt[-1+c*x]),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && LtQ[m,-1]

```

$$2: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m < -1$$

Derivation: Inverted integration by parts

Rule: If  $e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m < -1$ , then

$$\begin{aligned}
& \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \\
& \frac{(f x)^{m+1} (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n}{f (m+1)} - \frac{2 e p}{f^2 (m+1)} \int (f x)^{m+2} (d+e x^2)^{p-1} (a+b \operatorname{ArcSinh}[c x])^n dx - \\
& \frac{b c n d^{\operatorname{IntPart}[p]} (d+e x^2)^{\operatorname{FracPart}[p]}}{f (m+1) (1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int (f x)^{m+1} (1+c^2 x^2)^{p-\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx
\end{aligned}$$

Program code:

```

Int [(f_*x_)^m_*(d_+e_*x_^2)^p_.*(a_+b_*ArcSinh[c_*x_])^n_.,x_Symbol] :=
(f*x)^(m+1)*(d+e*x^2)^p*(a+b*ArcSinh[c*x])^n/(f*(m+1)) -
2*e*p/(f^2*(m+1))*Int[(f*x)^(m+2)*(d+e*x^2)^(p-1)*(a+b*ArcSinh[c*x])^n,x] -
b*c*n*d^IntPart[p]*(d+e*x^2)^FracPart[p]/(f*(m+1)*(1+c^2*x^2)^FracPart[p])*
Int[(f*x)^(m+1)*(1+c^2*x^2)^(p-1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[e,c^2*d] && GtQ[n,0] && GtQ[p,0] && LtQ[m,-1]

```

```

Int [(f_.**x_)^m_*(d1_+e1_.*x_)^p_*(d2_+e2_.*x_)^p_*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
(f**x)^(m+1)*(d1+e1*x)^p*(d2+e2*x)^p*(a+b*ArcCosh[c*x])^n/(f*(m+1)) -
2*e1*e2*p/(f^(2*(m+1))*Int[(f*x)^(m+2)*(d1+e1*x)^(p-1)*(d2+e2*x)^(p-1)*(a+b*ArcCosh[c*x])^n,x] -
b*c*n*(-d1*d2)^(p-1/2)*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]/(f*(m+1)*Sqrt[1+c*x]*Sqrt[-1+c*x])*
Int[(f**x)^(m+1)*(-1+c^2*x^2)^(p-1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && GtQ[p,0] && LtQ[m,-1] && IntegerQ[p-1/2]

```

$$2. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m \neq -1$$

$$1: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m \neq -1 \wedge (p \in \mathbb{Z} \vee d > 0)$$

Derivation: Inverted integration by parts

Rule: If  $e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m \neq -1 \wedge (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(f x)^{m+1} (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n}{f (m+2 p+1)} + \frac{2 d p}{m+2 p+1} \int (f x)^m (d+e x^2)^{p-1} (a+b \operatorname{ArcSinh}[c x])^n dx - \frac{b c n d^p}{f (m+2 p+1)} \int (f x)^{m+1} (1+c^2 x^2)^{p-\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

Program code:

```

(* Int [(f_.**x_)^m_*(d_+e_.*x_^2)^p_*(a_+b_.*ArcSinh[c_.*x_])^n_.,x_Symbol] :=
(f**x)^(m+1)*(d+e*x^2)^p*(a+b*ArcSinh[c*x])^n/(f*(m+2*p+1)) +
2*d*p/(m+2*p+1)*Int[(f*x)^m*(d+e*x^2)^(p-1)*(a+b*ArcSinh[c*x])^n,x] -
b*c*n*d^p/(f*(m+2*p+1))*Int[(f*x)^(m+1)*(1+c^2*x^2)^(p-1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[e,c^2*d] && GtQ[n,0] && GtQ[p,0] && Not[LtQ[m,-1]] && (IntegerQ[p] || GtQ[d,0]) && (RationalQ[m] || EqQ[n,1])

```

```

Int [(f_.**x_)^m_*(d_+e_.*x_^2)^p_*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
(f**x)^(m+1)*(d+e*x^2)^p*(a+b*ArcCosh[c*x])^n/(f*(m+2*p+1)) +
2*d*p/(m+2*p+1)*Int[(f*x)^m*(d+e*x^2)^(p-1)*(a+b*ArcCosh[c*x])^n,x] -
b*c*n*(-d)^p/(f*(m+2*p+1))*Int[(f*x)^(m+1)*(1+c*x)^(p-1/2)*(-1+c*x)^(p-1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[c^2*d+e,0] && GtQ[n,0] && GtQ[p,0] && Not[LtQ[m,-1]] && IntegerQ[p] && (RationalQ[m] || EqQ[n,1])

```

$$2. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m \neq -1$$

$$1: \int (f x)^m \sqrt{d+e x^2} (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge m \neq -1$$

Derivation: Inverted integration by parts

Note: The piecewise constant factor in the second integral reduces the degree of  $d$  in the resulting antiderivative.

Rule: If  $e = c^2 d \wedge n > 0 \wedge m \neq -1$ , then

$$\int (f x)^m \sqrt{d+e x^2} (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(f x)^{m+1} \sqrt{d+e x^2} (a+b \operatorname{ArcSinh}[c x])^n}{f (m+2)} - \frac{b c n \sqrt{d+e x^2}}{f (m+2) \sqrt{1+c^2 x^2}} \int (f x)^{m+1} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx + \frac{\sqrt{d+e x^2}}{(m+2) \sqrt{1+c^2 x^2}} \int \frac{(f x)^m (a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{1+c^2 x^2}} dx$$

Program code:

```
Int[(f_.*x_)^m_*Sqrt[d+_.*x_^2]*(a_+_.*b_.*ArcSinh[c_.*x_])^n_.,x_Symbol] :=
  (f*x)^(m+1)*Sqrt[d+e*x^2]*(a+b*ArcSinh[c*x])^n/(f*(m+2)) -
  b*c*n*Sqrt[d+e*x^2]/(f*(m+2)*Sqrt[1+c^2*x^2])*Int[(f*x)^(m+1)*(a+b*ArcSinh[c*x])^(n-1),x] +
  Sqrt[d+e*x^2]/((m+2)*Sqrt[1+c^2*x^2])*Int[(f*x)^m*(a+b*ArcSinh[c*x])^n/Sqrt[1+c^2*x^2],x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[e,c^2*d] && GtQ[n,0] && Not[LtQ[m,-1]] && (RationalQ[m] || EqQ[n,1])
```

```
Int[(f_.*x_)^m_*Sqrt[d1+_.*e1_.*x_]*Sqrt[d2+_.*e2_.*x_]*(a_+_.*b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  (f*x)^(m+1)*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]*(a+b*ArcCosh[c*x])^n/(f*(m+2)) -
  b*c*n*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]/(f*(m+2)*Sqrt[1+c*x]*Sqrt[-1+c*x])*Int[(f*x)^(m+1)*(a+b*ArcCosh[c*x])^(n-1),x] -
  Sqrt[d1+e1*x]*Sqrt[d2+e2*x]/((m+2)*Sqrt[1+c*x]*Sqrt[-1+c*x])*Int[(f*x)^m*(a+b*ArcCosh[c*x])^n/(Sqrt[1+c*x]*Sqrt[-1+c*x]),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && Not[LtQ[m,-1]] && (RationalQ[m] || EqQ[n,1])
```



2:  $\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx$  when  $e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m \neq -1$

Derivation: Inverted integration by parts

Rule: If  $e = c^2 d \wedge n > 0 \wedge p > 0 \wedge m \neq -1$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow$$

$$\frac{(f x)^{m+1} (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n}{f (m+2p+1)} + \frac{2 d p}{m+2p+1} \int (f x)^m (d+e x^2)^{p-1} (a+b \operatorname{ArcSinh}[c x])^n dx -$$

$$\frac{b c n d \operatorname{IntPart}[p] (d+e x^2)^{\operatorname{FracPart}[p]}}{f (m+2p+1) (1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int (f x)^{m+1} (1+c^2 x^2)^{p-\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

Program code:

```
Int[(f_*x_)^m_*(d+_e_*x_^2)^p_*(a+_b_*ArcSinh[c_*x_])^n_,x_Symbol] :=
  (f*x)^(m+1)*(d+e*x^2)^p*(a+b*ArcSinh[c*x])^n/(f*(m+2*p+1)) +
  2*d*p/(m+2*p+1)*Int[(f*x)^m*(d+e*x^2)^(p-1)*(a+b*ArcSinh[c*x])^n,x] -
  b*c*n*d*IntPart[p]*(d+e*x^2)^FracPart[p]/(f*(m+2*p+1)*(1+c^2*x^2)^FracPart[p])*
  Int[(f*x)^(m+1)*(1+c^2*x^2)^(p-1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[e,c^2*d] && GtQ[n,0] && GtQ[p,0] && Not[LtQ[m,-1]] && (RationalQ[m] || EqQ[n,1])
```

```
Int[(f_*x_)^m_*(d1+_e1_*x_)^p_*(d2+_e2_*x_)^p_*(a+_b_*ArcCosh[c_*x_])^n_,x_Symbol] :=
  (f*x)^(m+1)*(d1+e1*x)^p*(d2+e2*x)^p*(a+b*ArcCosh[c*x])^n/(f*(m+2*p+1)) +
  2*d1*d2*p/(m+2*p+1)*Int[(f*x)^m*(d1+e1*x)^(p-1)*(d2+e2*x)^(p-1)*(a+b*ArcCosh[c*x])^n,x] -
  b*c*n*(-d1*d2)^(p-1/2)*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]/(f*(m+2*p+1)*Sqrt[1+c*x]*Sqrt[-1+c*x])*
  Int[(f*x)^(m+1)*(-1+c^2*x^2)^(p-1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && GtQ[p,0] && Not[LtQ[m,-1]] &&
IntegerQ[p-1/2] && (RationalQ[m] || EqQ[n,1])
```

$$6. \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge m < -1 \wedge m \in \mathbb{Z}$$

$$1: \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge m < -1 \wedge m \in \mathbb{Z} \wedge (p \in \mathbb{Z} \vee d > 0)$$

Rule: If  $e = c^2 d \wedge n > 0 \wedge m < -1 \wedge m \in \mathbb{Z} \wedge (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(f x)^{m+1} (d + e x^2)^{p+1} (a + b \operatorname{ArcSinh}[c x])^n}{d f (m+1)} - \frac{c^2 (m+2p+3)}{f^2 (m+1)} \int (f x)^{m+2} (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx - \frac{b c n d^p}{f (m+1)} \int (f x)^{m+1} (1 + c^2 x^2)^{p+\frac{1}{2}} (a + b \operatorname{ArcSinh}[c x])^{n-1} dx$$

Programcode:

```
(* Int[(f.*x_)^m.*(d.+e.*x_^2)^p.*(a.+b.*ArcSinh[c.*x_])^n.,x_Symbol] :=
(f*x)^(m+1)*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n/(d*f*(m+1)) -
c^2*(m+2*p+3)/(f^2*(m+1))*Int[(f*x)^(m+2)*(d+e*x^2)^p*(a+b*ArcSinh[c*x])^n,x] -
b*c*n*d^p/(f*(m+1))*Int[(f*x)^(m+1)*(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,p},x] && EqQ[e,c^2*d] && GtQ[n,0] && LtQ[m,-1] && IntegerQ[m] && (IntegerQ[p] || GtQ[d,0]) *)
```

```
Int[(f.*x_)^m.*(d.+e.*x_^2)^p.*(a.+b.*ArcCosh[c.*x_])^n.,x_Symbol] :=
(f*x)^(m+1)*(d+e*x^2)^(p+1)*(a+b*ArcCosh[c*x])^n/(d*f*(m+1)) +
c^2*(m+2*p+3)/(f^2*(m+1))*Int[(f*x)^(m+2)*(d+e*x^2)^p*(a+b*ArcCosh[c*x])^n,x] +
b*c*n*(-d)^p/(f*(m+1))*Int[(f*x)^(m+1)*(1+c*x)^(p+1/2)*(-1+c*x)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,p},x] && EqQ[c^2*d+e,0] && GtQ[n,0] && LtQ[m,-1] && IntegerQ[m] && IntegerQ[p]
```

$$2: \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge m < -1 \wedge m \in \mathbb{Z}$$

Rule: If  $e = c^2 d \wedge n > 0 \wedge m < -1 \wedge m \in \mathbb{Z}$ , then

$$\int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow$$

$$\frac{(f x)^{m+1} (d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])^n}{d f (m+1)} - \frac{c^2 (m+2 p+3)}{f^2 (m+1)} \int (f x)^{m+2} (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx -$$

$$\frac{b c n d^{\operatorname{IntPart}[p]} (d+e x^2)^{\operatorname{FracPart}[p]}}{f (m+1) (1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int (f x)^{m+1} (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

## Programcode:

```
Int[(f_.**x_)^m_*(d_+e_.**x_^2)^p_*(a_+b_.**ArcSinh[c_.**x_] )^n_.,x_Symbol] :=
(f**x)^(m+1)*(d+e**x^2)^(p+1)*(a+b**ArcSinh[c**x])^n/(d*f*(m+1)) -
c^2*(m+2*p+3)/(f^2*(m+1))*Int[(f**x)^(m+2)*(d+e**x^2)^p*(a+b**ArcSinh[c**x])^n,x] -
b*c*n*d^IntPart[p]*(d+e**x^2)^FracPart[p]/(f*(m+1)*(1+c^2**x^2)^FracPart[p])*
Int[(f**x)^(m+1)*(1+c^2**x^2)^(p+1/2)*(a+b**ArcSinh[c**x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,p},x] && EqQ[e,c^2*d] && GtQ[n,0] && LtQ[m,-1] && IntegerQ[m]
```

```
Int[(f_.**x_)^m_*(d1_+e1_.**x_)^p_*(d2_+e2_.**x_)^p_*(a_+b_.**ArcCosh[c_.**x_] )^n_.,x_Symbol] :=
(f**x)^(m+1)*(d1+e1**x)^(p+1)*(d2+e2**x)^(p+1)*(a+b**ArcCosh[c**x])^n/(d1*d2*f*(m+1)) +
c^2*(m+2*p+3)/(f^2*(m+1))*Int[(f**x)^(m+2)*(d1+e1**x)^p*(d2+e2**x)^p*(a+b**ArcCosh[c**x])^n,x] +
b*c*n*(-d1*d2)^IntPart[p]*(d1+e1**x)^FracPart[p]*(d2+e2**x)^FracPart[p]/(f*(m+1)*(1+c**x)^FracPart[p]*(-1+c**x)^FracPart[p])*
Int[(f**x)^(m+1)*(-1+c^2**x^2)^(p+1/2)*(a+b**ArcCosh[c**x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && LtQ[m,-1] && IntegerQ[m] && IntegerQ[p+1/2]
```

```
Int[(f_.**x_)^m_*(d1_+e1_.**x_)^p_*(d2_+e2_.**x_)^p_*(a_+b_.**ArcCosh[c_.**x_] )^n_.,x_Symbol] :=
(f**x)^(m+1)*(d1+e1**x)^(p+1)*(d2+e2**x)^(p+1)*(a+b**ArcCosh[c**x])^n/(d1*d2*f*(m+1)) +
c^2*(m+2*p+3)/(f^2*(m+1))*Int[(f**x)^(m+2)*(d1+e1**x)^p*(d2+e2**x)^p*(a+b**ArcCosh[c**x])^n,x] +
b*c*n*(-d1*d2)^IntPart[p]*(d1+e1**x)^FracPart[p]*(d2+e2**x)^FracPart[p]/(f*(m+1)*(1+c**x)^FracPart[p]*(-1+c**x)^FracPart[p])*
Int[(f**x)^(m+1)*(1+c**x)^(p+1/2)*(-1+c**x)^(p+1/2)*(a+b**ArcCosh[c**x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && LtQ[m,-1] && IntegerQ[m]
```

$$7. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p < -1$$

$$1. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p < -1 \wedge m > 1$$

$$1: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p < -1 \wedge m > 1 \wedge (p \in \mathbb{Z} \vee d > 0)$$

## Derivation: Integration by parts

$$\text{Basis: } x (d+e x^2)^p = \partial_x \frac{(d+e x^2)^{p+1}}{2e(p+1)}$$

Rule: If  $e = c^2 d \wedge n > 0 \wedge p < -1 \wedge m > 1 \wedge (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{f (f x)^{m-1} (d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])^n}{2e(p+1)} - \frac{f^2(m-1)}{2e(p+1)} \int (f x)^{m-2} (d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])^n dx - \frac{b f n d^p}{2c(p+1)} \int (f x)^{m-1} (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

## Program code:

```
(* Int[(f.*x_)^m.*(d+e.*x_^2)^p.*(a.+b.*ArcSinh[c.*x_])^n.,x_Symbol] :=
  f*(f*x)^(m-1)*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n/(2*e*(p+1)) -
  f^2*(m-1)/(2*e*(p+1))*Int[(f*x)^(m-2)*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n,x] -
  b*f*n*d^p/(2*c*(p+1))*Int[(f*x)^(m-1)*(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[e,c^2*d] && GtQ[n,0] && LtQ[p,-1] && GtQ[m,1] && (IntegerQ[p] || GtQ[d,0]) *)
```

```
Int[(f.*x_)^m.*(d+e.*x_^2)^p.*(a.+b.*ArcCosh[c.*x_])^n.,x_Symbol] :=
  f*(f*x)^(m-1)*(d+e*x^2)^(p+1)*(a+b*ArcCosh[c*x])^n/(2*e*(p+1)) -
  f^2*(m-1)/(2*e*(p+1))*Int[(f*x)^(m-2)*(d+e*x^2)^(p+1)*(a+b*ArcCosh[c*x])^n,x] -
  b*f*n*(-d)^p/(2*c*(p+1))*Int[(f*x)^(m-1)*(1+c*x)^(p+1/2)*(-1+c*x)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[c^2*d+e,0] && GtQ[n,0] && LtQ[p,-1] && GtQ[m,1] && IntegerQ[p]
```

$$2: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p < -1 \wedge m > 1$$

Derivation: Integration by parts

$$\text{Basis: } x (d+e x^2)^p = \partial_x \frac{(d+e x^2)^{p+1}}{2e(p+1)}$$

Rule: If  $e = c^2 d \wedge n > 0 \wedge p < -1 \wedge m > 1$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow$$

$$\frac{f (f x)^{m-1} (d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])^n}{2e(p+1)} - \frac{f^2 (m-1)}{2e(p+1)} \int (f x)^{m-2} (d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])^n dx -$$

$$\frac{b f n d^{\operatorname{IntPart}[p]} (d+e x^2)^{\operatorname{FracPart}[p]}}{2c(p+1) (1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int (f x)^{m-1} (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

Program code:

```
Int[(f_*x_)^m_*(d_+e_*x_^2)^p_*(a_+b_*ArcSinh[c_*x_])^n_,x_Symbol] :=
  f*(f*x)^(m-1)*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n/(2*e*(p+1)) -
  f^2*(m-1)/(2*e*(p+1))*Int[(f*x)^(m-2)*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n,x] -
  b*f*n*d^IntPart[p]*(d+e*x^2)^FracPart[p]/(2*c*(p+1)*(1+c^2*x^2)^FracPart[p])*
  Int[(f*x)^(m-1)*(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[e,c^2*d] && GtQ[n,0] && LtQ[p,-1] && GtQ[m,1]
```

```
Int[(f_*x_)^m_*(d1_+e1_*x_)^p_*(d2_+e2_*x_)^p_*(a_+b_*ArcCosh[c_*x_])^n_,x_Symbol] :=
  f*(f*x)^(m-1)*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n/(2*e1*e2*(p+1)) -
  f^2*(m-1)/(2*e1*e2*(p+1))*Int[(f*x)^(m-2)*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n,x] -
  b*f*n*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(2*c*(p+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[(f*x)^(m-1)*(-1+c^2*x^2)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && LtQ[p,-1] && GtQ[m,1] && IntegerQ[p+1/2]
```

```

Int [(f_.*x_)^m_*(d1_+e1_.*x_)^p_*(d2_+e2_.*x_)^p_*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
f*(f*x)^(m-1)*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n/(2*e1*e2*(p+1)) -
f^2*(m-1)/(2*e1*e2*(p+1))*Int[(f*x)^(m-2)*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n,x] -
b*f*n*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(2*c*(p+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
Int[(f*x)^(m-1)*(1+c*x)^(p+1/2)*(-1+c*x)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && LtQ[p,-1] && Not[IntegerQ[p]] && GtQ[m,1]

```

$$2. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p < -1 \wedge m \neq 1$$

$$1: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p < -1 \wedge m \neq 1 \wedge (p \in \mathbb{Z} \vee d > 0)$$

Rule: If  $e = c^2 d \wedge n > 0 \wedge p < -1 \wedge m \neq 1 \wedge (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(f x)^{m+1} (d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])^n}{2 d f (p+1)} + \frac{m+2 p+3}{2 d (p+1)} \int (f x)^m (d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])^n dx + \frac{b c n d^p}{2 f (p+1)} \int (f x)^{m+1} (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

Program code:

```

(* Int [(f_.*x_)^m_*(d_+e_.*x_^2)^p_*(a_+b_.*ArcSinh[c_.*x_])^n_.,x_Symbol] :=
-(f*x)^(m+1)*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n/(2*d*f*(p+1)) +
(m+2*p+3)/(2*d*(p+1))*Int[(f*x)^m*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n,x] +
b*c*n*d^p/(2*f*(p+1))*Int[(f*x)^(m+1)*(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[e,c^2*d] && GtQ[n,0] && LtQ[p,-1] && Not[GtQ[m,1]] && (IntegerQ[p] || GtQ[d,0]) &&
(IntegerQ[m] || IntegerQ[p] || EqQ[n,1]) *)

```

```

Int [(f_.*x_)^m_*(d_+e_.*x_^2)^p_*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
-(f*x)^(m+1)*(d+e*x^2)^(p+1)*(a+b*ArcCosh[c*x])^n/(2*d*f*(p+1)) +
(m+2*p+3)/(2*d*(p+1))*Int[(f*x)^m*(d+e*x^2)^(p+1)*(a+b*ArcCosh[c*x])^n,x] -
b*c*n*(-d)^p/(2*f*(p+1))*Int[(f*x)^(m+1)*(1+c*x)^(p+1/2)*(-1+c*x)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[c^2*d+e,0] && GtQ[n,0] && LtQ[p,-1] && Not[GtQ[m,1]] && IntegerQ[p]

```

$$2: \int (f x)^m (d+e x^2)^p (a+b \operatorname{arcsinh}(c x))^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge p < -1 \wedge m \neq 1$$

Rule: If  $e = c^2 d \wedge n > 0 \wedge p < -1 \wedge m \neq 1$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{arcsinh}(c x))^n dx \rightarrow$$

$$-\frac{(f x)^{m+1} (d+e x^2)^{p+1} (a+b \operatorname{arcsinh}(c x))^n}{2 d f (p+1)} + \frac{m+2 p+3}{2 d (p+1)} \int (f x)^m (d+e x^2)^{p+1} (a+b \operatorname{arcsinh}(c x))^n dx +$$

$$\frac{b c n d^{\operatorname{IntPart}[p]} (d+e x^2)^{\operatorname{FracPart}[p]}}{2 f (p+1) (1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int (f x)^{m+1} (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{arcsinh}(c x))^{n-1} dx$$

Program code:

```
Int[(f_.**x_)^m_*(d_+e_.**x_^2)^p_*(a_.+b_.**ArcSinh[c_.**x_])^n_,x_Symbol] :=
-(f*x)^(m+1)*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n/(2*d*f*(p+1)) +
(m+2*p+3)/(2*d*(p+1))*Int[(f*x)^m*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n,x] +
b*c*n*d^IntPart[p]*(d+e*x^2)^FracPart[p]/(2*f*(p+1)*(1+c^2*x^2)^FracPart[p])*
Int[(f*x)^(m+1)*(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[e,c^2*d] && GtQ[n,0] && LtQ[p,-1] && Not[GtQ[m,1]] && (IntegerQ[m] || IntegerQ[p] || EqQ[n,1])
```

```
Int[(f_.**x_)^m_*(d1_+e1_.**x_)^p_*(d2_+e2_.**x_)^p_*(a_.+b_.**ArcCosh[c_.**x_])^n_,x_Symbol] :=
-(f*x)^(m+1)*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n/(2*d1*d2*f*(p+1)) +
(m+2*p+3)/(2*d1*d2*(p+1))*Int[(f*x)^m*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n,x] -
b*c*n*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(2*f*(p+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
Int[(f*x)^(m+1)*(-1+c^2*x^2)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && LtQ[p,-1] && Not[GtQ[m,1]] &&
(IntegerQ[m] || EqQ[n,1]) && IntegerQ[p+1/2]
```

```
Int[(f_.**x_)^m_*(d1_+e1_.**x_)^p_*(d2_+e2_.**x_)^p_*(a_.+b_.**ArcCosh[c_.**x_])^n_,x_Symbol] :=
-(f*x)^(m+1)*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n/(2*d1*d2*f*(p+1)) +
(m+2*p+3)/(2*d1*d2*(p+1))*Int[(f*x)^m*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n,x] -
b*c*n*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(2*f*(p+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
Int[(f*x)^(m+1)*(1+c*x)^(p+1/2)*(-1+c*x)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && LtQ[p,-1] && Not[GtQ[m,1]] &&
(IntegerQ[m] || IntegerQ[p] || EqQ[n,1])
```

$$8. \int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \text{ when } e = c^2 d \wedge n > 0$$

$$1. \int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \text{ when } e = c^2 d \wedge n > 0 \wedge m > 1$$

$$1: \int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \text{ when } e = c^2 d \wedge n > 0 \wedge m > 1 \wedge d > 0$$

Rule: If  $e = c^2 d \wedge n > 0 \wedge m > 1 \wedge d > 0$ , then

$$\int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \rightarrow \frac{f (f x)^{m-1} \sqrt{d + e x^2} (a + b \operatorname{ArcSinh}[c x])^n}{e m} - \frac{b f n}{c m \sqrt{d}} \int (f x)^{m-1} (a + b \operatorname{ArcSinh}[c x])^{n-1} dx - \frac{f^2 (m-1)}{c^2 m} \int \frac{(f x)^{m-2} (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx$$

Program code:

```
(* Int[(f_.*x_)^m_*(a_.+b_.*ArcSinh[c_.*x_])^n_/Sqrt[d_+e_.*x_^2],x_Symbol] :=
f*(f*x)^(m-1)*Sqrt[d+e*x^2]*(a+b*ArcSinh[c*x])^n/(e*m) -
b*f*n*Sqrt[1+c^2*x^2]/(c*m*Sqrt[d+e*x^2])*Int[(f*x)^(m-1)*(a+b*ArcSinh[c*x])^(n-1),x] -
f^2*(m-1)/(c^2*m)*Int[(f*x)^(m-2)*(a+b*ArcSinh[c*x])^n/Sqrt[d+e*x^2],x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[e,c^2*d] && GtQ[n,0] && GtQ[m,1] && GtQ[d,0] && IntegerQ[m] *)
```

```
(* Int[(f_.*x_)^m_*(a_.+b_.*ArcCosh[c_.*x_])^n_/ (Sqrt[d1_+e1_.*x_]*Sqrt[d2_+e2_.*x_]),x_Symbol] :=
f*(f*x)^(m-1)*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]*(a+b*ArcCosh[c*x])^n/(e1*e2*m) +
b*f*n*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]/(c*d1*d2*m*Sqrt[1+c*x]*Sqrt[-1+c*x])*Int[(f*x)^(m-1)*(a+b*ArcCosh[c*x])^(n-1),x] +
f^2*(m-1)/(c^2*m)*Int[(f*x)^(m-2)*(a+b*ArcCosh[c*x])^n/(Sqrt[d1+e1*x]*Sqrt[d2+e2*x]),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && GtQ[m,1] && IntegerQ[m] *)
```



$$2: \int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \text{ when } e = c^2 d \wedge n > 0 \wedge m > 1$$

Rule: If  $e = c^2 d \wedge n > 0 \wedge m > 1$ , then

$$\int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \rightarrow$$

$$\frac{f (f x)^{m-1} \sqrt{d + e x^2} (a + b \operatorname{ArcSinh}[c x])^n}{e m} -$$

$$\frac{b f n \sqrt{1 + c^2 x^2}}{c m \sqrt{d + e x^2}} \int (f x)^{m-1} (a + b \operatorname{ArcSinh}[c x])^{n-1} dx - \frac{f^2 (m-1)}{c^2 m} \int \frac{(f x)^{m-2} (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx$$

Program code:

```
Int[(f_.**x_)^m_*(a_.+b_.**ArcSinh[c_.**x_])^n_/Sqrt[d_+e_.**x_^2],x_Symbol] :=
f*(f**x)^(m-1)*Sqrt[d+e**x^2]*(a+b**ArcSinh[c**x])^n/(e*m) -
b*f*n*Sqrt[1+c^2*x^2]/(c*m*Sqrt[d+e*x^2])*Int[(f**x)^(m-1)*(a+b**ArcSinh[c**x])^(n-1),x] -
f^2*(m-1)/(c^2*m)*Int[(f**x)^(m-2)*(a+b**ArcSinh[c**x])^n/Sqrt[d+e*x^2],x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[e,c^2*d] && GtQ[n,0] && GtQ[m,1] && IntegerQ[m]
```

```
Int[(f_.**x_)^m_*(a_.+b_.**ArcCosh[c_.**x_])^n_/(Sqrt[d1_+e1_.**x_]*Sqrt[d2_+e2_.**x_]),x_Symbol] :=
f*(f**x)^(m-1)*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]*(a+b**ArcCosh[c**x])^n/(e1*e2*m) +
b*f*n*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]/(c*d1*d2*m*Sqrt[1+c*x]*Sqrt[-1+c*x])*Int[(f**x)^(m-1)*(a+b**ArcCosh[c**x])^(n-1),x] +
f^2*(m-1)/(c^2*m)*Int[(f**x)^(m-2)*(a+b**ArcCosh[c**x])^n/(Sqrt[d1+e1*x]*Sqrt[d2+e2*x]),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && GtQ[m,1] && IntegerQ[m]
```

$$2: \int \frac{x^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \text{ when } e = c^2 d \wedge d > 0 \wedge n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If  $e = c^2 d \wedge d > 0 \wedge m \in \mathbb{Z}$ , then  $\frac{x^m}{\sqrt{d+e x^2}} = \frac{1}{c^{m+1} \sqrt{d}} \operatorname{Subst}[\operatorname{Sinh}[x]^m, x, \operatorname{ArcSinh}[c x]] \partial_x \operatorname{ArcSinh}[c x]$

Note: If  $n \in \mathbb{Z}^+$ , then  $(a+b x)^n \operatorname{Sinh}[x]$  is integrable in closed-form.

Rule: If  $e = c^2 d \wedge d > 0 \wedge n \in \mathbb{Z}^+ \wedge m \in \mathbb{Z}$ , then

$$\int \frac{x^m (a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} dx \rightarrow \frac{1}{c^{m+1} \sqrt{d}} \operatorname{Subst}\left[\int (a+b x)^n \operatorname{Sinh}[x]^m dx, x, \operatorname{ArcSinh}[c x]\right]$$

Program code:

```
Int[x^m_*(a_.*b_.*ArcSinh[c_*x_])^n_/Sqrt[d_+e_*x_^2],x_Symbol] :=
  1/(c^(m+1)*Sqrt[d])*Subst[Int[(a+b*x)^n*Sinh[x]^m,x],x,ArcSinh[c*x]] /;
FreeQ[{a,b,c,d,e},x] && EqQ[e,c^2*d] && GtQ[d,0] && IGtQ[n,0] && IntegerQ[m]
```

```
Int[x^m_*(a_.*b_.*ArcCosh[c_*x_])^n_/(Sqrt[d1_+e1_*x_]*Sqrt[d2_+e2_*x_]),x_Symbol] :=
  1/(c^(m+1)*Sqrt[-d1*d2])*Subst[Int[(a+b*x)^n*Cosh[x]^m,x],x,ArcCosh[c*x]] /;
FreeQ[{a,b,c,d1,e1,d2,e2},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IGtQ[n,0] && GtQ[d1,0] && LtQ[d2,0] && IntegerQ[m]
```

$$3: \int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])}{\sqrt{d + e x^2}} dx \text{ when } e = c^2 d \wedge d > 0 \wedge m \notin \mathbb{Z}$$

Rule: If  $e = c^2 d \wedge d > 0 \wedge m \notin \mathbb{Z}$ , then

$$\int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])}{\sqrt{d + e x^2}} dx \rightarrow$$

$$\frac{(f x)^{m+1} (a + b \operatorname{ArcSinh}[c x]) \operatorname{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1+m}{2}, \frac{3+m}{2}, -c^2 x^2\right]}{\sqrt{d} f (m+1)} -$$

$$\frac{b c (f x)^{m+2} \operatorname{HypergeometricPFQ}\left[\{1, 1 + \frac{m}{2}, 1 + \frac{m}{2}\}, \left\{\frac{3}{2} + \frac{m}{2}, 2 + \frac{m}{2}\right\}, -c^2 x^2\right]}{\sqrt{d} f^2 (m+1) (m+2)}$$

Program code:

```
Int[(f_.**x_)^m_*(a_.+b_.*ArcSinh[c_.**x_])/Sqrt[d_+e_.**x_^2],x_Symbol] :=
  (f**x)^(m+1)*(a+b*ArcSinh[c*x])*Hypergeometric2F1[1/2,(1+m)/2,(3+m)/2,-c^2*x^2]/(Sqrt[d]*f*(m+1)) -
  b*c*(f**x)^(m+2)*HypergeometricPFQ[{1,1+m/2,1+m/2},{3/2+m/2,2+m/2},-c^2*x^2]/(Sqrt[d]*f^2*(m+1)*(m+2)) /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[e,c^2*d] && GtQ[d,0] && Not[IntegerQ[m]]
```

```
Int[(f_.**x_)^m_*(a_.+b_.*ArcCosh[c_.**x_])/(Sqrt[d1_+e1_.**x_]*Sqrt[d2_+e2_.**x_]),x_Symbol] :=
  (f**x)^(m+1)*Sqrt[1-c^2*x^2]*(a+b*ArcCosh[c*x])*Hypergeometric2F1[1/2,(1+m)/2,(3+m)/2,c^2*x^2]/
  (f*(m+1)*Sqrt[d1+e1*x]*Sqrt[d2+e2*x]) +
  b*c*(f**x)^(m+2)*HypergeometricPFQ[{1,1+m/2,1+m/2},{3/2+m/2,2+m/2},c^2*x^2]/(Sqrt[-d1*d2]*f^2*(m+1)*(m+2)) /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[d1,0] && LtQ[d2,0] && Not[IntegerQ[m]]
```

$$4: \int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \text{ when } e = c^2 d \wedge n > 0 \wedge d \neq 0$$

Derivation: Piecewise constant extraction

Basis: If  $e = c^2 d$ , then  $\partial_x \frac{\sqrt{1+c^2 x^2}}{\sqrt{d+e x^2}} = 0$

Rule: If  $e = c^2 d \wedge n > 0 \wedge d \neq 0$ , then

$$\int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \rightarrow \frac{\sqrt{1 + c^2 x^2}}{\sqrt{d + e x^2}} \int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{1 + c^2 x^2}} dx$$

Program code:

```
Int[(f_.*x_)^m_*(a_.+b_.*ArcSinh[c_.*x_])^n_/Sqrt[d_+e_.*x_^2],x_Symbol] :=
  Sqrt[1+c^2*x^2]/Sqrt[d+e*x^2]*Int[(f*x)^m*(a+b*ArcSinh[c*x])^n/Sqrt[1+c^2*x^2],x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[e,c^2*d] && GtQ[n,0] && Not[GtQ[d,0]] && (IntegerQ[m] || EqQ[n,1])
```

```
Int[(f_.*x_)^m_*(a_.+b_.*ArcCosh[c_.*x_])^n_/ (Sqrt[d1_+e1_.*x_]*Sqrt[d2_+e2_.*x_]),x_Symbol] :=
  Sqrt[1+c*x]*Sqrt[-1+c*x] / (Sqrt[d1+e1*x]*Sqrt[d2+e2*x])*Int[(f*x)^m*(a+b*ArcCosh[c*x])^n/ (Sqrt[1+c*x]*Sqrt[-1+c*x]),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && Not[GtQ[d1,0] && LtQ[d2,0]] && (IntegerQ[m] || EqQ[n,1])
```

$$9. \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge m > 1 \wedge m + 2p + 1 \neq 0$$

$$1: \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge m > 1 \wedge m + 2p + 1 \neq 0 \wedge (p \in \mathbb{Z} \vee d > 0)$$

Rule: If  $e = c^2 d \wedge n > 0 \wedge m > 1 \wedge m + 2p + 1 \neq 0 \wedge (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{f (f x)^{m-1} (d + e x^2)^{p+1} (a + b \operatorname{ArcSinh}[c x])^n}{e (m + 2p + 1)}$$

$$\frac{f^2 (m-1)}{c^2 (m+2p+1)} \int (f x)^{m-2} (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx - \frac{b f n d^p}{c (m+2p+1)} \int (f x)^{m-1} (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

### Program code:

```
(* Int[(f_.x_)^m_*(d_+e_.x_^2)^p_*(a_+b_.*ArcSinh[c_.x_])^n_,x_Symbol] :=
  f*(f*x)^(m-1)*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n/(e*(m+2*p+1)) -
  f^2*(m-1)/(c^2*(m+2*p+1))*Int[(f*x)^(m-2)*(d+e*x^2)^p*(a+b*ArcSinh[c*x])^n,x] -
  b*f*n*d^p/(c*(m+2*p+1))*Int[(f*x)^(m-1)*(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,p},x] && EqQ[e,c^2*d] && GtQ[n,0] && GtQ[m,1] && NeQ[m+2*p+1,0] && (IntegerQ[p] || GtQ[d,0]) && IntegerQ[m] *)
```

```
Int[(f_.x_)^m_*(d_+e_.x_^2)^p_*(a_+b_.*ArcCosh[c_.x_])^n_,x_Symbol] :=
  f*(f*x)^(m-1)*(d+e*x^2)^(p+1)*(a+b*ArcCosh[c*x])^n/(e*(m+2*p+1)) +
  f^2*(m-1)/(c^2*(m+2*p+1))*Int[(f*x)^(m-2)*(d+e*x^2)^p*(a+b*ArcCosh[c*x])^n,x] -
  b*f*n*(-d)^p/(c*(m+2*p+1))*Int[(f*x)^(m-1)*(1+c*x)^(p+1/2)*(-1+c*x)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,p},x] && EqQ[c^2*d+e,0] && GtQ[n,0] && GtQ[m,1] && NeQ[m+2*p+1,0] && IntegerQ[p] && IntegerQ[m]
```

$$2: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n > 0 \wedge m > 1 \wedge m+2p+1 \neq 0$$

Rule: If  $e = c^2 d \wedge n > 0 \wedge m > 1 \wedge m+2p+1 \neq 0$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow$$

$$\frac{f (f x)^{m-1} (d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])^n}{e (m+2p+1)} - \frac{f^2 (m-1)}{c^2 (m+2p+1)} \int (f x)^{m-2} (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx -$$

$$\frac{b f n d^{\operatorname{IntPart}[p]} (d+e x^2)^{\operatorname{FracPart}[p]}}{c (m+2p+1) (1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int (f x)^{m-1} (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n-1} dx$$

Program code:

```
Int[(f_.*x_)^m_*(d_+e_.*x_^2)^p_*(a_+b_.*ArcSinh[c_.*x_])^n_.,x_Symbol] :=
  f*(f*x)^(m-1)*(d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])^n/(e*(m+2*p+1)) -
  f^2*(m-1)/(c^2*(m+2*p+1))*Int[(f*x)^(m-2)*(d+e*x^2)^p*(a+b*ArcSinh[c*x])^n,x] -
  b*f*n*d^IntPart[p]*(d+e*x^2)^FracPart[p]/(c*(m+2*p+1)*(1+c^2*x^2)^FracPart[p])*
  Int[(f*x)^(m-1)*(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f,p},x] && EqQ[e,c^2*d] && GtQ[n,0] && GtQ[m,1] && NeQ[m+2*p+1,0] && IntegerQ[m]
```

```
Int[(f_.*x_)^m_*(d1_+e1_.*x_)^p_*(d2_+e2_.*x_)^p_*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  f*(f*x)^(m-1)*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n/(e1*e2*(m+2*p+1)) +
  f^2*(m-1)/(c^2*(m+2*p+1))*Int[(f*x)^(m-2)*(d1+e1*x)^p*(d2+e2*x)^p*(a+b*ArcCosh[c*x])^n,x] -
  b*f*n*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(c*(m+2*p+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[(f*x)^(m-1)*(-1+c^2*x^2)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && GtQ[m,1] && NeQ[m+2*p+1,0] && IntegerQ[m] && IntegerQ[p+1]
```

```
Int[(f_.*x_)^m_*(d1_+e1_.*x_)^p_*(d2_+e2_.*x_)^p_*(a_+b_.*ArcCosh[c_.*x_])^n_.,x_Symbol] :=
  f*(f*x)^(m-1)*(d1+e1*x)^(p+1)*(d2+e2*x)^(p+1)*(a+b*ArcCosh[c*x])^n/(e1*e2*(m+2*p+1)) +
  f^2*(m-1)/(c^2*(m+2*p+1))*Int[(f*x)^(m-2)*(d1+e1*x)^p*(d2+e2*x)^p*(a+b*ArcCosh[c*x])^n,x] -
  b*f*n*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(c*(m+2*p+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[(f*x)^(m-1)*(1+c*x)^(p+1/2)*(-1+c*x)^(p+1/2)*(a+b*ArcCosh[c*x])^(n-1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[n,0] && GtQ[m,1] && NeQ[m+2*p+1,0] && IntegerQ[m]
```

$$2. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n < -1$$

$$1. \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n < -1 \wedge m+2p+1 = 0$$

$$1: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n < -1 \wedge m+2p+1 = 0 \wedge (p \in \mathbb{Z} \vee d > 0)$$

### Derivation: Integration by parts

$$\text{Basis: } \frac{(a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{1+c^2 x^2}} = \partial_x \frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c (n+1)}$$

Rule: If  $e = c^2 d \wedge n < -1 \wedge m+2p+1 = 0 \wedge (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{d^p (f x)^m (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c (n+1)} - \frac{f m d^p}{b c (n+1)} \int (f x)^{m-1} (1+c^2 x^2)^{p-\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n+1} dx$$

### Program code:

```
(* Int[(f_*x_)^m_.*(d_+e_*x_^2)^p_.*(a_+b_*ArcSinh[c_*x_])^n_,x_Symbol] :=
d^p*(f*x)^m*(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n+1)/(b*c*(n+1)) -
f*m*d^p/(b*c*(n+1))*Int[(f*x)^(m-1)*(1+c^2*x^2)^(p-1/2)*(a+b*ArcSinh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f,m,p},x] && EqQ[e,c^2*d] && LtQ[n,-1] && EqQ[m+2*p+1,0] && (IntegerQ[p] || GtQ[d,0]) *)
```

```
Int[(f_*x_)^m_.*(d_+e_*x_^2)^p_.*(a_+b_*ArcCosh[c_*x_])^n_,x_Symbol] :=
(f*x)^m*Sqrt[1+c*x]*Sqrt[-1+c*x]*(d+e*x^2)^p*(a+b*ArcCosh[c*x])^(n+1)/(b*c*(n+1)) +
f*m*(-d)^p/(b*c*(n+1))*Int[(f*x)^(m-1)*(1+c*x)^(p-1/2)*(-1+c*x)^(p-1/2)*(a+b*ArcCosh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f,m,p},x] && EqQ[c^2*d+e,0] && LtQ[n,-1] && EqQ[m+2*p+1,0] && IntegerQ[p]
```

$$2: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n < -1 \wedge m+2p+1 = 0$$

Derivation: Integration by parts

$$\text{Basis: } \frac{(a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{1+c^2 x^2}} = \partial_x \frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c (n+1)}$$

Rule: If  $e = c^2 d \wedge n < -1 \wedge m+2p+1 = 0$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{(f x)^m \sqrt{1+c^2 x^2} (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c (n+1)} - \frac{f m d^{\operatorname{IntPart}[p]} (d+e x^2)^{\operatorname{FracPart}[p]}}{b c (n+1) (1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int (f x)^{m-1} (1+c^2 x^2)^{p-\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n+1} dx$$

Program code:

```
Int[(f_.**x_)^m_.*(d_+e_.**x_)^p_.*(a_+b_.*ArcSinh[c_.**x_])^n_,x_Symbol] :=
(f**x)^m*Sqrt[1+c^2*x^2]*(d+e*x^2)^p*(a+b*ArcSinh[c*x])^(n+1)/(b*c*(n+1)) -
f*m*d^IntPart[p]*(d+e*x^2)^FracPart[p]/(b*c*(n+1)*(1+c^2*x^2)^FracPart[p])*
Int[(f**x)^(m-1)*(1+c^2*x^2)^(p-1/2)*(a+b*ArcSinh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f,m,p},x] && EqQ[e,c^2*d] && LtQ[n,-1] && EqQ[m+2*p+1,0]
```

```
Int[(f_.**x_)^m_.*(d1_+e1_.**x_)^p_.*(d2_+e2_.**x_)^p_.*(a_+b_.*ArcCosh[c_.**x_])^n_,x_Symbol] :=
(f**x)^m*Sqrt[1+c*x]*Sqrt[-1+c*x]*(d1+e1*x)^p*(d2+e2*x)^p*(a+b*ArcCosh[c*x])^(n+1)/(b*c*(n+1)) +
f*m*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(b*c*(n+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
Int[(f**x)^(m-1)*(-1+c^2*x^2)^(p-1/2)*(a+b*ArcCosh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && LtQ[n,-1] && EqQ[m+2*p+1,0] && IntegerQ[p-1/2]
```



```

Int [(f_.*x_)^m_.*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_.*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
(f*x)^m*Sqrt[1+c*x]*Sqrt[-1+c*x]*(d1+e1*x)^p*(d2+e2*x)^p*(a+b*ArcCosh[c*x])^(n+1)/(b*c*(n+1)) +
f*m*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(b*c*(n+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
Int[(f*x)^(m-1)*(1+c*x)^(p-1/2)*(-1+c*x)^(p-1/2)*(a+b*ArcCosh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m,p},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && LtQ[n,-1] && EqQ[m+2*p+1,0]

```

2:  $\int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx$  when  $e = c^2 d \wedge n < -1 \wedge d > 0$

Derivation: Integration by parts

Basis: If  $e = c^2 d \wedge d > 0$ , then  $\frac{(a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} = \partial_x \frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c \sqrt{d} (n+1)}$

Rule: If  $e = c^2 d \wedge n < -1 \wedge d > 0$ , then

$$\int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \rightarrow \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^{n+1}}{b c \sqrt{d} (n+1)} - \frac{f m}{b c \sqrt{d} (n+1)} \int (f x)^{m-1} (a + b \operatorname{ArcSinh}[c x])^{n+1} dx$$

Program code:

```

Int [(f_.*x_)^m_.*(a_+b_.*ArcSinh[c_.*x_])^n_/Sqrt[d_+e_.*x_^2],x_Symbol] :=
(f*x)^m*(a+b*ArcSinh[c*x])^(n+1)/(b*c*Sqrt[d]*(n+1)) -
f*m/(b*c*Sqrt[d]*(n+1))*Int[(f*x)^(m-1)*(a+b*ArcSinh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[e,c^2*d] && LtQ[n,-1] && GtQ[d,0]

```

```

Int [(f_.*x_)^m_.*(a_+b_.*ArcCosh[c_.*x_])^n_/ (Sqrt[d1_+e1_.*x_]*Sqrt[d2_+e2_.*x_]),x_Symbol] :=
(f*x)^m*(a+b*ArcCosh[c*x])^(n+1)/(b*c*Sqrt[-d1+d2]*(n+1)) -
(f*m)/(b*c*Sqrt[-d1+d2]*(n+1))*Int[(f*x)^(m-1)*(a+b*ArcCosh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && LtQ[n,-1] && GtQ[d1,0] && LtQ[d2,0]

```

$$\mathbf{x:} \int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \text{ when } e = c^2 d \wedge n < -1 \wedge d \neq 0$$

Derivation: Piecewise constant extraction

$$\text{Basis: If } e = c^2 d, \text{ then } \partial_x \frac{\sqrt{1+c^2 x^2}}{\sqrt{d+e x^2}} = 0$$

Rule: If  $e = c^2 d \wedge n < -1 \wedge d \neq 0$ , then

$$\int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{d + e x^2}} dx \rightarrow \frac{\sqrt{1 + c^2 x^2}}{\sqrt{d + e x^2}} \int \frac{(f x)^m (a + b \operatorname{ArcSinh}[c x])^n}{\sqrt{1 + c^2 x^2}} dx$$

Program code:

```
(* Int[(f_.x_)^m_.*(a_.+b_.*ArcSinh[c_.x_])^n_/Sqrt[d_+e_.x_^2],x_Symbol] :=
  Sqrt[1+c^2*x^2]/Sqrt[d+e*x^2]*Int[(f*x)^m*(a+b*ArcSinh[c*x])^n/Sqrt[1+c^2*x^2],x] /;
FreeQ[{a,b,c,d,e,f,m},x] && EqQ[e,c^2*d] && LtQ[n,-1] && Not[GtQ[d,0]] *)
```

```
(* Int[(f_.x_)^m_.*(a_.+b_.*ArcCosh[c_.x_])^n_/ (Sqrt[d1_+e1_.x_]*Sqrt[d2_+e2_.x_]),x_Symbol] :=
  Sqrt[1+c*x]*Sqrt[-1+c*x]/(Sqrt[d1+e1*x]*Sqrt[d2+e2*x])*Int[(f*x)^m*(a+b*ArcCosh[c*x])^n/(Sqrt[1+c*x]*Sqrt[-1+c*x]),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f,m},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && LtQ[n,-1] && Not[GtQ[d1,0] && LtQ[d2,0]] *)
```

$$3. \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n < -1 \wedge m + 3 \in \mathbb{Z}^+ \wedge 2p \in \mathbb{Z}^+$$

$$\mathbf{1:} \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge n < -1 \wedge m + 3 \in \mathbb{Z}^+ \wedge 2p \in \mathbb{Z}^+ \wedge (p \in \mathbb{Z} \vee d > 0)$$

Derivation: Integration by parts

$$\text{Basis: } \frac{(a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{1+c^2 x^2}} = \partial_x \frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c (n+1)}$$

Rule: If  $e = c^2 d \wedge n < -1 \wedge m + 3 \in \mathbb{Z}^+ \wedge 2p \in \mathbb{Z}^+ \wedge (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow$$

$$\frac{d^p (f x)^m (1+c^2 x^2)^{p+\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c (n+1)} -$$

$$\frac{f m d^p}{b c (n+1)} \int (f x)^{m-1} (1+c^2 x^2)^{p-\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n+1} dx - \frac{c d^p (m+2p+1)}{b f (n+1)} \int (f x)^{m+1} (1+c^2 x^2)^{p-\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n+1} dx$$

### Program code:

```
(* Int[(f.*x_)^m.*(d+e.*x_^2)^p.*(a.+b.*ArcSinh[c.*x_])^n,x_Symbol] :=
d^p*(f*x)^m*(1+c^2*x^2)^(p+1/2)*(a+b*ArcSinh[c*x])^(n+1)/(b*c*(n+1)) -
f*m*d^p/(b*c*(n+1))*Int[(f*x)^(m-1)*(1+c^2*x^2)^(p-1/2)*(a+b*ArcSinh[c*x])^(n+1),x] -
c*d^p*(m+2*p+1)/(b*f*(n+1))*Int[(f*x)^(m+1)*(1+c^2*x^2)^(p-1/2)*(a+b*ArcSinh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[e,c^2*d] && LtQ[n,-1] && IGtQ[m,-3] && IGtQ[2*p,0] && (IntegerQ[p] || GtQ[d,0]) *)
```

```
Int[(f.*x_)^m.*(d+e.*x_^2)^p.*(a.+b.*ArcCosh[c.*x_])^n,x_Symbol] :=
(f*x)^m*Sqrt[1+c*x]*Sqrt[-1+c*x]*(d+e*x^2)^p*(a+b*ArcCosh[c*x])^(n+1)/(b*c*(n+1)) +
f*m*(-d)^p/(b*c*(n+1))*Int[(f*x)^(m-1)*(1+c*x)^(p-1/2)*(-1+c*x)^(p-1/2)*(a+b*ArcCosh[c*x])^(n+1),x] -
c*(-d)^p*(m+2*p+1)/(b*f*(n+1))*Int[(f*x)^(m+1)*(1+c*x)^(p-1/2)*(-1+c*x)^(p-1/2)*(a+b*ArcCosh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[c^2*d+e,0] && LtQ[n,-1] && IGtQ[m,-3] && IGtQ[p,0]
```

**2:**  $\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx$  when  $e = c^2 d \wedge n < -1 \wedge m+3 \in \mathbb{Z}^+ \wedge 2p \in \mathbb{Z}^+$

### Derivation: Integration by parts

**Basis:**  $\frac{(a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{1+c^2 x^2}} = \partial_x \frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c (n+1)}$

**Rule:** If  $e = c^2 d \wedge n < -1 \wedge m+3 \in \mathbb{Z}^+ \wedge 2p \in \mathbb{Z}^+$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow$$

$$\frac{(f x)^m \sqrt{1+c^2 x^2} (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c (n+1)} -$$

$$\frac{f m d^{\operatorname{IntPart}[p]} (d+e x^2)^{\operatorname{FracPart}[p]}}{b c (n+1) (1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int (f x)^{m-1} (1+c^2 x^2)^{p-\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n+1} dx -$$

$$\frac{c (m+2 p+1) d^{\operatorname{IntPart}[p]} (d+e x^2)^{\operatorname{FracPart}[p]}}{b f (n+1) (1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int (f x)^{m+1} (1+c^2 x^2)^{p-\frac{1}{2}} (a+b \operatorname{ArcSinh}[c x])^{n+1} dx$$

### Program code:

```
Int[(f_.**x_)^m_.*(d_+e_.**x_^2)^p_.*(a_+b_.**ArcSinh[c_.**x_])^n_,x_Symbol] :=
(f**x)^m*Sqrt[1+c^2*x^2]*(d+e*x^2)^p*(a+b*ArcSinh[c*x])^(n+1)/(b*c*(n+1)) -
f*m*d^IntPart[p]*(d+e*x^2)^FracPart[p]/(b*c*(n+1)*(1+c^2*x^2)^FracPart[p])*
Int[(f**x)^(m-1)*(1+c^2*x^2)^(p-1/2)*(a+b*ArcSinh[c*x])^(n+1),x] -
c*(m+2*p+1)*d^IntPart[p]*(d+e*x^2)^FracPart[p]/(b*f*(n+1)*(1+c^2*x^2)^FracPart[p])*
Int[(f**x)^(m+1)*(1+c^2*x^2)^(p-1/2)*(a+b*ArcSinh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[e,c^2*d] && LtQ[n,-1] && IGtQ[m,-3] && IGtQ[2*p,0]
```

```
Int[(f_.**x_)^m_.*(d1_+e1_.**x_)^p_.*(d2_+e2_.**x_)^p_.*(a_+b_.**ArcCosh[c_.**x_])^n_,x_Symbol] :=
(f**x)^m*Sqrt[1+c*x]*Sqrt[-1+c*x]*(d1+e1*x)^p*(d2+e2*x)^p*(a+b*ArcCosh[c*x])^(n+1)/(b*c*(n+1)) +
f*m*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(b*c*(n+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
Int[(f**x)^(m-1)*(-1+c^2*x^2)^(p-1/2)*(a+b*ArcCosh[c*x])^(n+1),x] -
c*(m+2*p+1)*(-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/(b*f*(n+1)*(1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
Int[(f**x)^(m+1)*(-1+c^2*x^2)^(p-1/2)*(a+b*ArcCosh[c*x])^(n+1),x] /;
FreeQ[{a,b,c,d1,e1,d2,e2,f},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && LtQ[n,-1] && IGtQ[m,-3] && IGtQ[p+1/2,0]
```

$$3. \int x^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge 2p \in \mathbb{Z} \wedge p > -1 \wedge m \in \mathbb{Z}^+$$

$$1: \int x^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e = c^2 d \wedge 2p \in \mathbb{Z} \wedge p > -1 \wedge m \in \mathbb{Z}^+ \wedge (p \in \mathbb{Z} \vee d > 0)$$

Derivation: Integration by substitution

$$\text{Basis: } F[x] = \frac{1}{c} \operatorname{Subst}\left[F\left[\frac{\operatorname{Sinh}[x]}{c}\right] \operatorname{Cosh}[x], x, \operatorname{ArcSinh}[c x]\right] \partial_x \operatorname{ArcSinh}[c x]$$

$$\text{Basis: If } e = c^2 d \wedge m \in \mathbb{Z} \wedge (p \in \mathbb{Z} \vee d > 0), \text{ then } x^m (d+e x^2)^p = \frac{d^p}{c^{m+1}} \operatorname{Subst}\left[\operatorname{Sinh}[x]^m \operatorname{Cosh}[x]^{2p+1}, x, \operatorname{ArcSinh}[c x]\right] \partial_x \operatorname{ArcSinh}[c x]$$

Rule: If  $e = c^2 d \wedge 2p \in \mathbb{Z} \wedge p > -1 \wedge m \in \mathbb{Z}^+ \wedge (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int x^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{d^p}{c^{m+1}} \operatorname{Subst}\left[\int (a+b x)^n \operatorname{Sinh}[x]^m \operatorname{Cosh}[x]^{2p+1} dx, x, \operatorname{ArcSinh}[c x]\right]$$

Program code:

```
Int[x^m.*(d+e.*x^2)^p.*(a.+b.*ArcSinh[c.*x])^n.,x_Symbol] :=
  d^p/c^(m+1)*Subst[Int[(a+b*x)^n*Sinh[x]^m*Cosh[x]^(2*p+1),x],x,ArcSinh[c*x] ] /;
FreeQ[{a,b,c,d,e,n},x] && EqQ[e,c^2*d] && IntegerQ[2*p] && GtQ[p,-1] && IGtQ[m,0] && (IntegerQ[p] || GtQ[d,0])
```

```
Int[x^m.*(d+e.*x^2)^p.*(a.+b.*ArcCosh[c.*x])^n.,x_Symbol] :=
  (-d)^p/c^(m+1)*Subst[Int[(a+b*x)^n*Cosh[x]^m*Sinh[x]^(2*p+1),x],x,ArcCosh[c*x] ] /;
FreeQ[{a,b,c,d,e,n},x] && EqQ[c^2*d+e,0] && IGtQ[p,0] && IGtQ[m,0]
```

```
Int[x^m.*(d1+e1.*x)^p.*(d2+e2.*x)^p.*(a.+b.*ArcCosh[c.*x])^n.,x_Symbol] :=
  (-d1*d2)^p/c^(m+1)*Subst[Int[(a+b*x)^n*Cosh[x]^m*Sinh[x]^(2*p+1),x],x,ArcCosh[c*x] ] /;
FreeQ[{a,b,c,d1,e1,d2,e2,n},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IntegerQ[p+1/2] && GtQ[p,-1] && IGtQ[m,0] && (GtQ[d1,0] && LtQ[d2,0])
```

**2:**  $\int x^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx$  when  $e = c^2 d \wedge 2p \in \mathbb{Z} \wedge p > -1 \wedge m \in \mathbb{Z}^+ \wedge \neg (p \in \mathbb{Z} \vee d > 0)$

Derivation: Piecewise constant extraction

Basis: If  $e = c^2 d$ , then  $\partial_x \frac{(d+e x^2)^p}{(1+c^2 x^2)^p} = 0$

Rule: If  $e = c^2 d \wedge 2p \in \mathbb{Z} \wedge p > -1 \wedge m \in \mathbb{Z}^+ \wedge \neg (p \in \mathbb{Z} \vee d > 0)$ , then

$$\int x^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{d^{\operatorname{IntPart}[p]} (d+e x^2)^{\operatorname{FracPart}[p]}}{(1+c^2 x^2)^{\operatorname{FracPart}[p]}} \int x^m (1+c^2 x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx$$

Program code:

```
Int[x_^m_.*(d_+e_.*x_^2)^p_.*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  d^IntPart[p]*(d+e*x^2)^FracPart[p]/(1+c^2*x^2)^FracPart[p]*Int[x^m*(1+c^2*x^2)^p*(a+b*ArcSinh[c*x])^n,x] /;
  FreeQ[{a,b,c,d,e,n},x] && EqQ[e,c^2*d] && IntegerQ[2*p] && GtQ[p,-1] && IGtQ[m,0] && Not[(IntegerQ[p] || GtQ[d,0])]
```

```
Int[x_^m_.*(d1_+e1_.*x_)^p_.*(d2_+e2_.*x_)^p_.*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  (-d1*d2)^IntPart[p]*(d1+e1*x)^FracPart[p]*(d2+e2*x)^FracPart[p]/((1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[x^m*(1+c*x)^p*(-1+c*x)^p*(a+b*ArcCosh[c*x])^n,x] /;
  FreeQ[{a,b,c,d1,e1,d2,e2,n},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && IntegerQ[2*p] && GtQ[p,-1] && IGtQ[m,0] &&
  Not[(IntegerQ[p] || GtQ[d1,0] && LtQ[d2,0])]
```

4:  $\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx$  when  $e = c^2 d \wedge d > 0 \wedge p + \frac{1}{2} \in \mathbb{Z}^+ \wedge \frac{m+1}{2} \notin \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If  $e = c^2 d \wedge d > 0 \wedge p + \frac{1}{2} \in \mathbb{Z}^+ \wedge \frac{m+1}{2} \notin \mathbb{Z}^+$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int \frac{(a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{d+e x^2}} \operatorname{ExpandIntegrand}[(f x)^m (d+e x^2)^{p+\frac{1}{2}}, x] dx$$

Program code:

```
Int[(f_.**x_)^m_*(d+_e_.**x_^2)^p_*(a+_b_.**ArcSinh[c_.**x_] )^n_.,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*ArcSinh[c*x])^n/Sqrt[d+e*x^2], (f*x)^m*(d+e*x^2)^(p+1/2),x],x] /;
  FreeQ[{a,b,c,d,e,f,m,n},x] && EqQ[e,c^2*d] && GtQ[d,0] && IGtQ[p+1/2,0] && Not[IGtQ[(m+1)/2,0]] && (EqQ[m,-1] || EqQ[m,-2])
```

```
Int[(f_.**x_)^m_*(d1+_e1_.**x_)^p_*(d2+_e2_.**x_)^p_*(a+_b_.**ArcCosh[c_.**x_] )^n_.,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*ArcCosh[c*x])^n/(Sqrt[d1+e1*x]*Sqrt[d2+e2*x]), (f*x)^m*(d1+e1*x)^(p+1/2)*(d2+e2*x)^(p+1/2),x],x] /;
  FreeQ[{a,b,c,d1,e1,d2,e2,f,m,n},x] && EqQ[e1-c*d1,0] && EqQ[e2+c*d2,0] && GtQ[d1,0] && LtQ[d2,0] && IGtQ[p+1/2,0] && Not[IGtQ[(m+1)/2,0]] &&
  (EqQ[m,-1] || EqQ[m,-2])
```

2.  $\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x])^n dx$  when  $e \neq c^2 d$

0:  $\int (f x)^m (d+e x^2) (a+b \operatorname{ArcCosh}[c x]) dx$  when  $c^2 d + e \neq 0 \wedge m \neq -1 \wedge m \neq -3$

Derivation: Integration by parts

Note: This rule can be removed when integrands of the form  $(d+e x)^m (f+g x)^m (a+c x^2)^p$  when  $e f + d g = 0$  are integrated without first resorting to piecewise constant extraction.

Rule: If  $c^2 d + e \neq 0 \wedge m \neq -1 \wedge m \neq -3$ , then

$$\int (f x)^m (d+e x^2) (a+b \operatorname{ArcCosh}[c x]) dx \rightarrow \frac{d (f x)^{m+1} (a+b \operatorname{ArcCosh}[c x])}{f (m+1)} + \frac{e (f x)^{m+3} (a+b \operatorname{ArcCosh}[c x])}{f^3 (m+3)} - \frac{b c}{f (m+1) (m+3)} \int \frac{(f x)^{m+1} (d (m+3) + e (m+1) x^2)}{\sqrt{1+c x} \sqrt{-1+c x}} dx$$

Program code:

```
Int[(f_*x_)^m_.*(d_+e_*x_^2)*(a_+b_*ArcCosh[c_*x_]),x_Symbol] :=
  d*(f*x)^(m+1)*(a+b*ArcCosh[c*x])/(f*(m+1)) +
  e*(f*x)^(m+3)*(a+b*ArcCosh[c*x])/(f^3*(m+3)) -
  b*c/(f*(m+1)*(m+3))*Int[(f*x)^(m+1)*(d*(m+3)+e*(m+1)*x^2)/(Sqrt[1+c*x]*Sqrt[-1+c*x]),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[c^2*d+e,0] && NeQ[m,-1] && NeQ[m,-3]
```

1:  $\int x (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx$  when  $e \neq c^2 d \wedge p \neq -1$

Derivation: Integration by parts

Basis:: If  $p \neq -1$ , then  $x (d+e x^2)^p = \partial_x \frac{(d+e x^2)^{p+1}}{2e(p+1)}$

Rule: If  $e \neq c^2 d \wedge p \neq -1$ , then

$$\int x (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \rightarrow \frac{(d+e x^2)^{p+1} (a+b \operatorname{ArcSinh}[c x])}{2e(p+1)} - \frac{b c}{2e(p+1)} \int \frac{(d+e x^2)^{p+1}}{\sqrt{1+c^2 x^2}} dx$$

Program code:

```
Int[x_*(d_+e_*x_^2)^p_.*(a_+b_*ArcSinh[c_*x_]),x_Symbol] :=
  (d+e*x^2)^(p+1)*(a+b*ArcSinh[c*x])/(2*e*(p+1)) - b*c/(2*e*(p+1))*Int[(d+e*x^2)^(p+1)/Sqrt[1+c^2*x^2],x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[e,c^2*d] && NeQ[p,-1]
```

```
Int[x_*(d_+e_*x_^2)^p_.*(a_+b_*ArcCosh[c_*x_]),x_Symbol] :=
  (d+e*x^2)^(p+1)*(a+b*ArcCosh[c*x])/(2*e*(p+1)) - b*c/(2*e*(p+1))*Int[(d+e*x^2)^(p+1)/(Sqrt[1+c*x]*Sqrt[-1+c*x]),x] /;
FreeQ[{a,b,c,d,e,p},x] && NeQ[c^2*d+e,0] && NeQ[p,-1]
```



$$2: \int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \text{ when } e \neq c^2 d \wedge p \in \mathbb{Z} \wedge (p > 0 \vee \frac{m-1}{2} \in \mathbb{Z}^+ \wedge m+p \leq 0)$$

Derivation: Integration by parts

Note: If  $\frac{m-1}{2} \in \mathbb{Z}^+ \wedge p \in \mathbb{Z}^- \wedge m+p \geq 0$ , then  $\int (f x)^m (d+e x^2)^p$  is a rational function.

Rule: If  $e \neq c^2 d \wedge p \in \mathbb{Z} \wedge (p > 0 \vee \frac{m-1}{2} \in \mathbb{Z}^+ \wedge m+p \leq 0)$ , let  $u = \int (f x)^m (d+e x^2)^p dx$ , then

$$\int (f x)^m (d+e x^2)^p (a+b \operatorname{ArcSinh}[c x]) dx \rightarrow u (a+b \operatorname{ArcSinh}[c x]) - b c \int \frac{u}{\sqrt{1+c^2 x^2}} dx$$

Program code:

```
Int[(f_.*x_)^m_.*(d_+e_.*x_^2)^p_.*(a_+b_.*ArcSinh[c_.*x_]),x_Symbol] :=
  With[{u=IntHide[(f*x)^m*(d+e*x^2)^p,x]},
    Dist[a+b*ArcSinh[c*x],u,x] - b*c*Int[SimplifyIntegrand[u/Sqrt[1+c^2*x^2],x],x] /;
    FreeQ[{a,b,c,d,e,f,m},x] && NeQ[e,c^2*d] && IntegerQ[p] && (GtQ[p,0] || IGtQ[(m-1)/2,0] && LeQ[m+p,0])
```

```
Int[(f_.*x_)^m_.*(d_+e_.*x_^2)^p_.*(a_+b_.*ArcCosh[c_.*x_]),x_Symbol] :=
  With[{u=IntHide[(f*x)^m*(d+e*x^2)^p,x]},
    Dist[a+b*ArcCosh[c*x],u,x] - b*c*Int[SimplifyIntegrand[u/(Sqrt[1+c*x]*Sqrt[-1+c*x]),x],x] /;
    FreeQ[{a,b,c,d,e,f,m},x] && NeQ[c^2*d+e,0] && IntegerQ[p] && (GtQ[p,0] || IGtQ[(m-1)/2,0] && LeQ[m+p,0])
```

$$3: \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } e \neq c^2 d \wedge n \in \mathbb{Z}^+ \wedge p \in \mathbb{Z} \wedge m \in \mathbb{Z}$$

### Derivation: Algebraic expansion

Rule: If  $e \neq c^2 d \wedge n \in \mathbb{Z}^+ \wedge p \in \mathbb{Z} \wedge m \in \mathbb{Z}$ , then

$$\int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int (a + b \operatorname{ArcSinh}[c x])^n \operatorname{ExpandIntegrand}[(f x)^m (d + e x^2)^p, x] dx$$

### Program code:

```
Int[(f_.**x_)^m_.*(d_+e_.**x_^2)^p_.*(a_+b_.*ArcSinh[c_.**x_])^n_.,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*ArcSinh[c*x])^n,(f*x)^m*(d+e*x^2)^p,x],x] /;
  FreeQ[{a,b,c,d,e,f},x] && NeQ[e,c^2*d] && IGtQ[n,0] && IntegerQ[p] && IntegerQ[m]
```

```
Int[(f_.**x_)^m_.*(d_+e_.**x_^2)^p_.*(a_+b_.*ArcCosh[c_.**x_])^n_.,x_Symbol] :=
  Int[ExpandIntegrand[(a+b*ArcCosh[c*x])^n,(f*x)^m*(d+e*x^2)^p,x],x] /;
  FreeQ[{a,b,c,d,e,f},x] && NeQ[c^2+d+e,0] && IGtQ[n,0] && IntegerQ[p] && IntegerQ[m]
```

$$x: \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$$

Rule:

$$\int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \int (f x)^m (d + e x^2)^p (a + b \operatorname{ArcSinh}[c x])^n dx$$

### Program code:

```
Int[(f_.**x_)^m_.*(d_+e_.**x_^2)^p_.*(a_+b_.*ArcSinh[c_.**x_])^n_.,x_Symbol] :=
  Unintegrable[(f*x)^m*(d+e*x^2)^p*(a+b*ArcSinh[c*x])^n,x] /;
  FreeQ[{a,b,c,d,e,f,m,n,p},x]
```

```
Int [(f_.**x_)^m_.*(d_+e_.**x_^2)^p_.*(a_+b_.*ArcCosh[c_.**x_])^n_.,x_Symbol] :=
  Unintegrable[(f*x)^m*(d+e*x^2)^p*(a+b*ArcCosh[c*x])^n,x] /;
  FreeQ[{a,b,c,d,e,f,m,n,p},x] && IntegerQ[p]
```

```
Int [(f_.**x_)^m_.*(d1_+e1_.**x_)^p_.*(d2_+e2_.**x_)^q_.*(a_+b_.*ArcCosh[c_.**x_])^n_.,x_Symbol] :=
  Unintegrable[(f*x)^m*(d1+e1*x)^p*(d2+e2*x)^q*(a+b*ArcCosh[c*x])^n,x] /;
  FreeQ[{a,b,c,d1,e1,d2,e2,f,m,n,p},x]
```

### Rules for integrands of the form $(h x)^m (d+e x)^p (f+g x)^q (a+b \operatorname{ArcSinh}[c x])^n$

1:  $\int (h x)^m (d+e x)^p (f+g x)^q (a+b \operatorname{ArcSinh}[c x])^n dx$  when  $e f + d g = 0 \wedge c^2 d^2 + e^2 = 0 \wedge (p | q) \in \mathbb{Z} + \frac{1}{2} \wedge p - q \geq 0 \wedge d > 0 \wedge \frac{g}{e} < 0$

#### Derivation: Algebraic expansion

Basis: If  $e f + d g = 0 \wedge c^2 d^2 + e^2 = 0 \wedge d > 0 \wedge \frac{g}{e} < 0$ , then

$$(d+e x)^p (f+g x)^q = \left(-\frac{d^2 g}{e}\right)^q (d+e x)^{p-q} (1+c^2 x^2)^q$$

Rule: If  $e f + d g = 0 \wedge c^2 d^2 + e^2 = 0 \wedge (p | q) \in \mathbb{Z} + \frac{1}{2} \wedge p - q \geq 0 \wedge d > 0 \wedge \frac{g}{e} < 0$ , then

$$\int (h x)^m (d+e x)^p (f+g x)^q (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \left(-\frac{d^2 g}{e}\right)^q \int (h x)^m (d+e x)^{p-q} (1+c^2 x^2)^q (a+b \operatorname{ArcSinh}[c x])^n dx$$

#### Program code:

```
Int [(h_.**x_)^m_.*(d_+e_.**x_)^p_.*(f_+g_.**x_)^q_.*(a_+b_.*ArcSinh[c_.**x_])^n_.,x_Symbol] :=
  (-d^2*g/e)^q*Int[(h*x)^m*(d+e*x)^(p-q)*(1+c^2*x^2)^q*(a+b*ArcSinh[c*x])^n,x] /;
  FreeQ[{a,b,c,d,e,f,g,h,m,n},x] && EqQ[e*f+d*g,0] && EqQ[c^2*d^2+e^2,0] && HalfIntegerQ[p,q] && GeQ[p-q,0] && GtQ[d,0] && LtQ[g/e,0]
```

$$2: \int (h x)^m (d+e x)^p (f+g x)^q (a+b \operatorname{ArcSinh}[c x])^n dx \text{ when } e f+d g==0 \wedge c^2 d^2+e^2==0 \wedge (p|q) \in \mathbb{Z}+\frac{1}{2} \wedge p-q \geq 0 \wedge \neg (d>0 \wedge \frac{g}{e}<0)$$

Derivation: Piecewise constant extraction

$$\text{Basis: If } e f+d g==0 \wedge c^2 d^2+e^2==0, \text{ then } \partial_x \frac{(d+e x)^q (f+g x)^q}{(1+c^2 x^2)^q} == 0$$

Rule: If  $e f+d g==0 \wedge c^2 d^2+e^2==0 \wedge (p|q) \in \mathbb{Z}+\frac{1}{2} \wedge p-q \geq 0 \wedge \neg (d>0 \wedge \frac{g}{e}<0)$ , then

$$\int (h x)^m (d+e x)^p (f+g x)^q (a+b \operatorname{ArcSinh}[c x])^n dx \rightarrow \frac{\left(-\frac{d+g}{e}\right)^{\operatorname{IntPart}[q]} (d+e x)^{\operatorname{FracPart}[q]} (f+g x)^{\operatorname{FracPart}[q]}}{(1+c^2 x^2)^{\operatorname{FracPart}[q]}} \int (h x)^m (d+e x)^{p-q} (1+c^2 x^2)^q (a+b \operatorname{ArcSinh}[c x])^n dx$$

Program code:

```
Int[(h_*x_)^m_*(d+e_*x_)^p_*(f+g_*x_)^q_*(a_+b_*ArcSinh[c_*x_])^n_,x_Symbol] :=
  (-d^2*g/e)^IntPart[q]*(d+e*x)^FracPart[q]*(f+g*x)^FracPart[q]/(1+c^2*x^2)^FracPart[q]*
  Int[(h*x)^m*(d+e*x)^(p-q)*(1+c^2*x^2)^q*(a+b*ArcSinh[c*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n},x] && EqQ[e*f+d*g,0] && EqQ[c^2*d^2+e^2,0] && HalfIntegerQ[p,q] && GeQ[p-q,0]
```

```
Int[(f_*x_)^m_*(d+e_*x_^2)^p_*(a_+b_*ArcCosh[c_*x_])^n_,x_Symbol] :=
  (-d)^IntPart[p]*(d+e*x^2)^FracPart[p]/((1+c*x)^FracPart[p]*(-1+c*x)^FracPart[p])*
  Int[(f*x)^m*(1+c*x)^p*(-1+c*x)^p*(a+b*ArcCosh[c*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && EqQ[c^2*d+e,0] && Not[IntegerQ[p]]
```