

## Rules for integrands of the form $(d x)^m (a + b \operatorname{ArcSinh}[c x])^n$

1.  $\int (d x)^m (a + b \operatorname{ArcSinh}[c x])^n dx$  when  $n \in \mathbb{Z}^+$

1:  $\int \frac{(a + b \operatorname{ArcSinh}[c x])^n}{x} dx$  when  $n \in \mathbb{Z}^+$

Derivation: Integration by substitution

Basis:  $\frac{1}{x} = \operatorname{Subst}\left[\frac{1}{\operatorname{Tanh}[x]}, x, \operatorname{ArcSinh}[c x]\right] \partial_x \operatorname{ArcSinh}[c x]$

Note:  $\frac{(a+bx)^n}{\operatorname{Tanh}[x]}$  is not integrable unless  $n \in \mathbb{Z}^+$ .

Rule: If  $n \in \mathbb{Z}^+$ , then

$$\int \frac{(a + b \operatorname{ArcSinh}[c x])^n}{x} dx \rightarrow \operatorname{Subst}\left[\int \frac{(a + b x)^n}{\operatorname{Tanh}[x]} dx, x, \operatorname{ArcSinh}[c x]\right]$$

Program code:

```
Int[(a_.+b_.*ArcSinh[c_.*x_])^n_/x_,x_Symbol] :=
  Subst[Int[(a+b*x)^n/Tanh[x],x],x,ArcSinh[c*x]] /;
  FreeQ[{a,b,c},x] && IGtQ[n,0]
```

```
Int[(a_.+b_.*ArcCosh[c_.*x_])^n_/x_,x_Symbol] :=
  Subst[Int[(a+b*x)^n/Coth[x],x],x,ArcCosh[c*x]] /;
  FreeQ[{a,b,c},x] && IGtQ[n,0]
```

2:  $\int (d x)^m (a + b \operatorname{ArcSinh}[c x])^n dx$  when  $n \in \mathbb{Z}^+ \wedge m \neq -1$

Derivation: Integration by parts

Rule: If  $n \in \mathbb{Z}^+ \wedge m \neq -1$ , then

$$\int (dx)^m (a+b \operatorname{ArcSinh}[cx])^n dx \rightarrow \frac{(dx)^{m+1} (a+b \operatorname{ArcSinh}[cx])^n}{d(m+1)} - \frac{bcn}{d(m+1)} \int \frac{(dx)^{m+1} (a+b \operatorname{ArcSinh}[cx])^{n-1}}{\sqrt{1+c^2 x^2}} dx$$

### Program code:

```
Int[(d.*x_)^m.*(a_.+b_.*ArcSinh[c.*x_])^n_,x_Symbol] :=
  (d*x)^(m+1)*(a+b*ArcSinh[c*x])^n/(d*(m+1)) -
  b*c*n/(d*(m+1))*Int[(d*x)^(m+1)*(a+b*ArcSinh[c*x])^(n-1)/Sqrt[1+c^2*x^2],x] /;
FreeQ[{a,b,c,d,m},x] && IGtQ[n,0] && NeQ[m,-1]
```

```
Int[(d.*x_)^m.*(a_.+b_.*ArcCosh[c.*x_])^n_,x_Symbol] :=
  (d*x)^(m+1)*(a+b*ArcCosh[c*x])^n/(d*(m+1)) -
  b*c*n/(d*(m+1))*Int[(d*x)^(m+1)*(a+b*ArcCosh[c*x])^(n-1)/(Sqrt[-1+c*x]*Sqrt[1+c*x]),x] /;
FreeQ[{a,b,c,d,m},x] && IGtQ[n,0] && NeQ[m,-1]
```

2.  $\int x^m (a+b \operatorname{ArcSinh}[cx])^n dx$  when  $m \in \mathbb{Z}^+$

**1:**  $\int x^m (a+b \operatorname{ArcSinh}[cx])^n dx$  when  $m \in \mathbb{Z}^+ \wedge n > 0$

### Derivation: Integration by parts

Rule: If  $n \in \mathbb{Z}^+ \wedge m \neq -1$ , then

$$\int x^m (a+b \operatorname{ArcSinh}[cx])^n dx \rightarrow \frac{x^{m+1} (a+b \operatorname{ArcSinh}[cx])^n}{m+1} - \frac{bcn}{m+1} \int \frac{x^{m+1} (a+b \operatorname{ArcSinh}[cx])^{n-1}}{\sqrt{1+c^2 x^2}} dx$$

### Program code:

```
Int[x^m.*(a_.+b_.*ArcSinh[c.*x_])^n_,x_Symbol] :=
  x^(m+1)*(a+b*ArcSinh[c*x])^n/(m+1) -
  b*c*n/(m+1)*Int[x^(m+1)*(a+b*ArcSinh[c*x])^(n-1)/Sqrt[1+c^2*x^2],x] /;
FreeQ[{a,b,c},x] && IGtQ[m,0] && GtQ[n,0]
```

```
Int[x^m_.*(a_.*b_.*ArcCosh[c_*x_])^n_,x_Symbol] :=  
  x^(m+1)*(a+b*ArcCosh[c*x])^n/(m+1) -  
  b*c*n/(m+1)*Int[x^(m+1)*(a+b*ArcCosh[c*x])^(n-1)/(Sqrt[-1+c*x]*Sqrt[1+c*x]),x] /;  
FreeQ[{a,b,c},x] && IGtQ[m,0] && GtQ[n,0]
```

$$2. \int x^m (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } m \in \mathbb{Z}^+ \wedge n < -1$$

$$1: \int x^m (a + b \operatorname{ArcSinh}[c x])^n dx \text{ when } m \in \mathbb{Z}^+ \wedge -2 \leq n < -1$$

Derivation: Integration by parts and integration by substitution

$$\text{Basis: } \frac{(a+b \operatorname{ArcSinh}[c x])^n}{\sqrt{1+c^2 x^2}} = \partial_x \frac{(a+b \operatorname{ArcSinh}[c x])^{n+1}}{b c (n+1)}$$

$$\text{Basis: } \frac{F[x]}{\sqrt{1+c^2 x^2}} = \frac{1}{c} F\left[\frac{\operatorname{Sinh}[\operatorname{ArcSinh}[c x]]}{c}\right] \partial_x \operatorname{ArcSinh}[c x]$$

Basis: If  $c > 0 \vee m \in \mathbb{Z}$ , then

$$\frac{x^{m-1} (m + (m+1) c^2 x^2)}{\sqrt{1+c^2 x^2}} = \frac{1}{c^m} \operatorname{Subst}\left[\operatorname{Sinh}[x]^{m-1} (m + (m+1) \operatorname{Sinh}[x]^2), x, \operatorname{ArcSinh}[c x]\right] \partial_x \operatorname{ArcSinh}[c x]$$

Note: Although not essential, by switching to the hyperbolic trig world this rule saves numerous steps and results in more compact antiderivatives.

Rule: If  $m \in \mathbb{Z}^+ \wedge -2 \leq n < -1$ , then

$$\begin{aligned} & \int x^m (a + b \operatorname{ArcSinh}[c x])^n dx \rightarrow \\ & \frac{x^m \sqrt{1+c^2 x^2} (a + b \operatorname{ArcSinh}[c x])^{n+1}}{b c (n+1)} - \frac{1}{b c (n+1)} \int \frac{x^{m-1} (m + (m+1) c^2 x^2) (a + b \operatorname{ArcSinh}[c x])^{n+1}}{\sqrt{1+c^2 x^2}} dx \rightarrow \\ & \frac{x^m \sqrt{1+c^2 x^2} (a + b \operatorname{ArcSinh}[c x])^{n+1}}{b c (n+1)} - \frac{1}{b c^{m+1} (n+1)} \operatorname{Subst}\left[\int (a + b x)^{n+1} \operatorname{Sinh}[x]^{m-1} (m + (m+1) \operatorname{Sinh}[x]^2) dx, x, \operatorname{ArcSinh}[c x]\right] \end{aligned}$$

Program code:

```
Int[x^m_.*(a_.+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  x^m*Sqrt[1+c^2*x^2]*(a+b*ArcSinh[c*x])^(n+1)/(b*c*(n+1)) -
  1/(b*c^(m+1)*(n+1))*Subst[Int[ExpandTrigReduce[(a+b*x)^(n+1),Sinh[x]^(m-1)*(m+(m+1)*Sinh[x]^2),x],x],x,ArcSinh[c*x]] /;
FreeQ[{a,b,c},x] && IGtQ[m,0] && GeQ[n,-2] && LtQ[n,-1]
```

```
Int[x_^m_.*(a_+b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  x^m*Sqrt[-1+c*x]*Sqrt[1+c*x]*(a+b*ArcCosh[c*x])^(n+1)/(b*c*(n+1)) +
  1/(b*c^(m+1)*(n+1))*Subst[Int[ExpandTrigReduce[(a+b*x)^(n+1)*Cosh[x]^(m-1)*(m-(m+1)*Cosh[x]^2),x],x],x,ArcCosh[c*x]] /;
FreeQ[{a,b,c},x] && IGtQ[m,0] && GeQ[n,-2] && LtQ[n,-1]
```

2:  $\int x^m (a + b \operatorname{ArcSinh}[cx])^n dx$  when  $m \in \mathbb{Z}^+ \wedge n < -2$

Derivation: Integration by parts and algebraic expansion

$$\text{Basis: } \frac{(a+b \operatorname{ArcSinh}[cx])^n}{\sqrt{1+c^2 x^2}} \equiv \partial_x \frac{(a+b \operatorname{ArcSinh}[cx])^{n+1}}{bc(n+1)}$$

$$\text{Basis: } \partial_x \left( x^m \sqrt{1+c^2 x^2} \right) \equiv \frac{m x^{m-1}}{\sqrt{1+c^2 x^2}} + \frac{c^2 (m+1) x^{m+1}}{\sqrt{1+c^2 x^2}}$$

Rule: If  $m \in \mathbb{Z}^+ \wedge n < -2$ , then

$$\int x^m (a + b \operatorname{ArcSinh}[cx])^n dx \rightarrow \frac{x^m \sqrt{1+c^2 x^2} (a + b \operatorname{ArcSinh}[cx])^{n+1}}{bc(n+1)} - \frac{m}{bc(n+1)} \int \frac{x^{m-1} (a + b \operatorname{ArcSinh}[cx])^{n+1}}{\sqrt{1+c^2 x^2}} dx - \frac{c(m+1)}{b(n+1)} \int \frac{x^{m+1} (a + b \operatorname{ArcSinh}[cx])^{n+1}}{\sqrt{1+c^2 x^2}} dx$$

Program code:

```
Int[x_^m_.*(a_+b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  x^m*Sqrt[1+c^2*x^2]*(a+b*ArcSinh[c*x])^(n+1)/(b*c*(n+1)) -
  m/(b*c*(n+1))*Int[x^(m-1)*(a+b*ArcSinh[c*x])^(n+1)/Sqrt[1+c^2*x^2],x] -
  c*(m+1)/(b*(n+1))*Int[x^(m+1)*(a+b*ArcSinh[c*x])^(n+1)/Sqrt[1+c^2*x^2],x] /;
FreeQ[{a,b,c},x] && IGtQ[m,0] && LtQ[n,-2]
```

```

Int [x_^m_.*(a_.*b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  x^m*Sqrt[-1+c*x]*Sqrt[1+c*x]*(a+b*ArcCosh[c*x])^(n+1)/(b*c*(n+1)) +
  m/(b*c*(n+1))*Int[x^(m-1)*(a+b*ArcCosh[c*x])^(n+1)/(Sqrt[-1+c*x]*Sqrt[1+c*x]),x] -
  c*(m+1)/(b*(n+1))*Int[x^(m+1)*(a+b*ArcCosh[c*x])^(n+1)/(Sqrt[-1+c*x]*Sqrt[1+c*x]),x] /;
FreeQ[{a,b,c},x] && IGtQ[m,0] && LtQ[n,-2]

```

3:  $\int x^m (a + b \operatorname{ArcSinh}[cx])^n dx$  when  $m \in \mathbb{Z}^+$

Derivation: Integration by substitution

Basis:  $F[x] = \frac{1}{c} \operatorname{Subst}\left[F\left[\frac{\operatorname{Sinh}[x]}{c}\right] \operatorname{Cosh}[x], x, \operatorname{ArcSinh}[cx]\right] \partial_x \operatorname{ArcSinh}[cx]$

Note: If  $m \in \mathbb{Z}^+$ , then  $(a + bx)^n \operatorname{Sinh}[x]^m \operatorname{Cosh}[x]$  is integrable in closed-form.

Rule: If  $m \in \mathbb{Z}^+$ , then

$$\int x^m (a + b \operatorname{ArcSinh}[cx])^n dx \rightarrow \frac{1}{c^{m+1}} \operatorname{Subst}\left[\int (a + bx)^n \operatorname{Sinh}[x]^m \operatorname{Cosh}[x] dx, x, \operatorname{ArcSinh}[cx]\right]$$

Program code:

```

Int [x_^m_.*(a_.*b_.*ArcSinh[c_.*x_])^n_,x_Symbol] :=
  1/c^(m+1)*Subst[Int[(a+b*x)^n*Sinh[x]^m*Cosh[x],x],x,ArcSinh[c*x]] /;
FreeQ[{a,b,c,n},x] && IGtQ[m,0]

```

```

Int [x_^m_.*(a_.*b_.*ArcCosh[c_.*x_])^n_,x_Symbol] :=
  1/c^(m+1)*Subst[Int[(a+b*x)^n*Cosh[x]^m*Sinh[x],x],x,ArcCosh[c*x]] /;
FreeQ[{a,b,c,n},x] && IGtQ[m,0]

```

**X:**  $\int (dx)^m (a + b \operatorname{ArcSinh}[cx])^n dx$

Rule:

$$\int (dx)^m (a + b \operatorname{ArcSinh}[cx])^n dx \rightarrow \int (dx)^m (a + b \operatorname{ArcSinh}[cx])^n dx$$

Program code:

```
Int[(d.*x_)^m.*(a.+b.*ArcSinh[c.*x_])^n.,x_Symbol] :=
  Unintegrable[(d*x)^m*(a+b*ArcSinh[c*x])^n,x] /;
  FreeQ[{a,b,c,d,m,n},x]
```

```
Int[(d.*x_)^m.*(a.+b.*ArcCosh[c.*x_])^n.,x_Symbol] :=
  Unintegrable[(d*x)^m*(a+b*ArcCosh[c*x])^n,x] /;
  FreeQ[{a,b,c,d,m,n},x]
```