

Rules for normalizing algebraic functions

$$2: \int x^m (e (a + b x^n)^r)^p (f (c + d x^n)^s)^q dx$$

Derivation: Piecewise constant extraction

$$\text{Basis: } \partial_x \frac{(e (a + b x^n)^r)^p (f (c + d x^n)^s)^q}{(a + b x^n)^{p r} (c + d x^n)^{q s}} = 0$$

Rule 1.5.4.2:

$$\int x^m (e (a + b x^n)^r)^p (f (c + d x^n)^s)^q dx \rightarrow \frac{(e (a + b x^n)^r)^p (f (c + d x^n)^s)^q}{(a + b x^n)^{p r} (c + d x^n)^{q s}} \int x^m (a + b x^n)^{p r} (c + d x^n)^{q s} dx$$

Program code:

```
Int[x^m_.*(e_.*(a_+b_.*x_^n_)^r_)^p_*(f_.*(c_+d_.*x_^n_)^s_)^q_,x_Symbol] :=
(e_*(a_+b_*x^n)^r)^p*(f_*(c_+d_*x^n)^s)^q/((a_+b_*x^n)^(p*r)*(c_+d_*x^n)^(q*s))*
Int[x^m*(a_+b_*x^n)^(p*r)*(c_+d_*x^n)^(q*s),x] /;
FreeQ[{a,b,c,d,e,f,m,n,p,q,r,s},x]
```

$$3. \int u \left(e \frac{a + b x^n}{c + d x^n} \right)^p dx$$

$$1: \int u \left(e \frac{a + b x^n}{c + d x^n} \right)^p dx \text{ when } b c - a d = 0$$

Derivation: Algebraic simplification

Basis: If $b c - a d = 0$, then $\frac{a+bz}{c+dz} = \frac{b}{d}$

Rule 1.5.4.3.1: If $b c - a d = 0$, then

$$\int u \left(e \frac{a + b x^n}{c + d x^n} \right)^p dx \rightarrow \left(\frac{b e}{d} \right)^p \int u dx$$

Program code:

```
Int[u.*(e.*(a_.+b_.*x_^n_.)/(c_.+d_.*x_^n_.))^p_,x_Symbol] :=
  (b*e/d)^p*Int[u,x] /;
  FreeQ[{a,b,c,d,e,n,p},x] && EqQ[b*c-a*d,0]
```

$$2. \int u \left(e^{\frac{a+bx^n}{c+dx^n}} \right)^p dx \text{ when } bc - ad \neq 0$$

$$1: \int u \left(e^{\frac{a+bx^n}{c+dx^n}} \right)^p dx \text{ when } bde > 0 \wedge c < \frac{ad}{b}$$

Derivation: Algebraic simplification

$$\text{Basis: If } bde > 0 \wedge \frac{ad}{b} \leq c, \text{ then } \left(e^{\frac{a+bz}{c+dz}} \right)^p == \frac{(e^{(a+bz)})^p}{(c+dz)^p}$$

Rule 1.5.4.3.2.1: If $bde > 0 \wedge c < \frac{ad}{b}$, then

$$\int u \left(e^{\frac{a+bx^n}{c+dx^n}} \right)^p dx \rightarrow \int \frac{u (e^{(a+bx^n)})^p}{(c+dx^n)^p} dx$$

Program code:

```
Int[u.*(e.*(a+.b.*x^n.)/(c+d.*x^n.))^p,x_Symbol] :=
  Int[u*(e*(a+b*x^n))^p/(c+d*x^n)^p,x] /;
  FreeQ[{a,b,c,d,e,n,p},x] && GtQ[b*d*e,0] && GtQ[c-a*d/b,0]
```

$$2. \int u \left(e^{\frac{a+bx^n}{c+dx^n}} \right)^p dx \text{ when } \neg (bde > 0 \wedge \frac{ad}{b} \leq c)$$

$$1: \int \left(e^{\frac{a+bx^n}{c+dx^n}} \right)^p dx \text{ when } \frac{1}{n} \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: If } \frac{1}{n} \in \mathbb{Z} \wedge q \in \mathbb{Z}^+, \text{ then } \left(e^{\frac{a+bx^n}{c+dx^n}} \right)^p == \frac{qe^{(bc-ad)}}{n} \text{ Subst} \left[\frac{x^{q(p+1)-1} (-ae+cx^q)^{\frac{1}{n}-1}}{(be-dx^q)^{\frac{1}{n}+1}}, x, \left(e^{\frac{a+bx^n}{c+dx^n}} \right)^{1/q} \right] \partial_x \left(e^{\frac{a+bx^n}{c+dx^n}} \right)^{1/q}$$

Rule 1.5.4.3.2.2.1: If $\frac{1}{n} \in \mathbb{Z}$, let $q = \text{Denominator}[p]$, then

$$\int \left(e \frac{a + b x^n}{c + d x^n} \right)^p dx \rightarrow \frac{q e (b c - a d)}{n} \text{Subst} \left[\int \frac{x^{q(p+1)-1} (-a e + c x^q)^{\frac{1}{n}-1}}{(b e - d x^q)^{\frac{1}{n}+1}} dx, x, \left(e \frac{a + b x^n}{c + d x^n} \right)^{1/q} \right]$$

Program code:

```
Int [ (e_.*(a_.*b_.*x_^n_.) / (c_.*d_.*x_^n_.) ) ^p_, x_Symbol ] :=
  With [ {q=Denominator[p]},
    q*e*(b*c-a*d) / n*Subst [
      Int [ x^(q*(p+1)-1) * (-a*e+c*x^q)^(1/n-1) / (b*e-d*x^q)^(1/n+1), x ], x, (e*(a+b*x^n) / (c+d*x^n))^(1/q) ] ] /;
  FreeQ[{a,b,c,d,e},x] && FractionQ[p] && IntegerQ[1/n]
```

2: $\int x^m \left(e \frac{a + b x^n}{c + d x^n} \right)^p dx$ when $\frac{m+1}{n} \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $\frac{m+1}{n} \in \mathbb{Z} \wedge q \in \mathbb{Z}^+$, then

$$x^m \left(e \frac{a + b x^n}{c + d x^n} \right)^p = \frac{q e (b c - a d)}{n} \text{Subst} \left[\frac{x^{q(p+1)-1} (-a e + c x^q)^{\frac{m+1}{n}-1}}{(b e - d x^q)^{\frac{m+1}{n}+1}}, x, \left(e \frac{a + b x^n}{c + d x^n} \right)^{1/q} \right] \partial_x \left(e \frac{a + b x^n}{c + d x^n} \right)^{1/q}$$

Rule 1.5.4.3.2.2.2: If $\frac{m+1}{n} \in \mathbb{Z}$, let $q = \text{Denominator}[p]$, then

$$\int x^m \left(e \frac{a + b x^n}{c + d x^n} \right)^p dx \rightarrow \frac{q e (b c - a d)}{n} \text{Subst} \left[\int \frac{x^{q(p+1)-1} (-a e + c x^q)^{\frac{m+1}{n}-1}}{(b e - d x^q)^{\frac{m+1}{n}+1}} dx, x, \left(e \frac{a + b x^n}{c + d x^n} \right)^{1/q} \right]$$

Program code:

```
Int [ x^m_.*(e_.*(a_.*b_.*x_^n_.) / (c_.*d_.*x_^n_.) ) ^p_, x_Symbol ] :=
  With [ {q=Denominator[p]},
    q*e*(b*c-a*d) / n*Subst [
      Int [ x^(q*(p+1)-1) * (-a*e+c*x^q)^(Simplify[(m+1)/n]-1) / (b*e-d*x^q)^(Simplify[(m+1)/n]+1), x ], x, (e*(a+b*x^n) / (c+d*x^n))^(1/q) ] ] /;
  FreeQ[{a,b,c,d,e,m,n},x] && FractionQ[p] && IntegerQ[Simplify[(m+1)/n]]
```

$$3: \int P_x^r \left(e \frac{a + b x^n}{c + d x^n} \right)^p dx \text{ when } \frac{1}{n} \in \mathbb{Z} \wedge r \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $\frac{1}{n} \in \mathbb{Z} \wedge q \in \mathbb{Z}^+$, then

$$F[x] \left(e \frac{a + b x^n}{c + d x^n} \right)^p == \frac{q e (b c - a d)}{n} \text{Subst} \left[\frac{x^{q(p+1)-1} (-a e + c x^q)^{\frac{1}{n}-1}}{(b e - d x^q)^{\frac{1}{n}+1}} F \left[\frac{(-a e + c x^q)^{\frac{1}{n}}}{(b e - d x^q)^{\frac{1}{n}}} \right], x, \left(e \frac{a + b x^n}{c + d x^n} \right)^{1/q} \right] \partial_x \left(e \frac{a + b x^n}{c + d x^n} \right)^{1/q}$$

Rule 1.5.4.3.2.2.3: If $\frac{1}{n} \in \mathbb{Z}$, let $q = \text{Denominator}[p]$, then

$$\int P_x^r \left(e \frac{a + b x^n}{c + d x^n} \right)^p dx \rightarrow \frac{q e (b c - a d)}{n} \text{Subst} \left[\int \frac{x^{q(p+1)-1} (-a e + c x^q)^{\frac{1}{n}-1}}{(b e - d x^q)^{\frac{1}{n}+1}} \text{Subst} \left[P_x, x, \frac{(-a e + c x^q)^{\frac{1}{n}}}{(b e - d x^q)^{\frac{1}{n}}} \right]^r dx, x, \left(e \frac{a + b x^n}{c + d x^n} \right)^{1/q} \right]$$

Program code:

```
Int[u^r.*(e.*(a.+b.*x^n_.)/(c.+d.*x^n_.))^p,x_Symbol] :=
  With[{q=Denominator[p]},
    q*e*(b*c-a*d)/n*Subst[Int[SimplifyIntegrand[x^(q*(p+1)-1)*(-a*e+c*x^q)^(1/n-1)/(b*e-d*x^q)^(1/n+1)*
      ReplaceAll[u,x(-a*e+c*x^q)^(1/n)/(b*e-d*x^q)^(1/n)]^r,x],x],x,(e*(a+b*x^n)/(c+d*x^n))^(1/q)] /;
    FreeQ[{a,b,c,d,e},x] && PolynomialQ[u,x] && FractionQ[p] && IntegerQ[1/n] && IntegerQ[r]
```

$$4: \int x^m P_x^r \left(e \frac{a + b x^n}{c + d x^n} \right)^p dx \text{ when } \frac{1}{n} \in \mathbb{Z} \wedge (m | r) \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $\frac{1}{n} \in \mathbb{Z} \wedge m \in \mathbb{Z} \wedge q \in \mathbb{Z}^+$, then

$$x^m F[x] \left(e \frac{a + b x^n}{c + d x^n} \right)^p == \frac{q e (b c - a d)}{n} \text{Subst} \left[\frac{x^q (p+1)^{-1} (-a e + c x^q)^{\frac{m+1}{n}-1}}{(b e - d x^q)^{\frac{m+1}{n}+1}} F \left[\frac{(-a e + c x^q)^{\frac{1}{n}}}{(b e - d x^q)^{\frac{1}{n}}} \right], x, \left(e \frac{a + b x^n}{c + d x^n} \right)^{1/q} \right] \partial_x \left(e \frac{a + b x^n}{c + d x^n} \right)^{1/q}$$

Rule 1.5.4.3.2.2.4: If $\frac{1}{n} \in \mathbb{Z} \wedge (m | r) \in \mathbb{Z}$, let $q = \text{Denominator}[p]$, then

$$\int x^m P_x^r \left(e \frac{a + b x^n}{c + d x^n} \right)^p dx \rightarrow \frac{q e (b c - a d)}{n} \text{Subst} \left[\int \frac{x^q (p+1)^{-1} (-a e + c x^q)^{\frac{m+1}{n}-1}}{(b e - d x^q)^{\frac{m+1}{n}+1}} \text{Subst} \left[P_x, x, \frac{(-a e + c x^q)^{\frac{1}{n}}}{(b e - d x^q)^{\frac{1}{n}}} \right]^r dx, x, \left(e \frac{a + b x^n}{c + d x^n} \right)^{1/q} \right]$$

Program code:

```
Int[x^m_*u^r_*(e_*(a_+b_*x^n_)/(c_+d_*x^n_))^p_,x_Symbol] :=
  With[{q=Denominator[p]},
    q*e*(b*c-a*d)/n*Subst[Int[SimplifyIntegrand[x^(q*(p+1)-1)*(-a*e+c*x^q)^(m+1/n-1)/(b*e-d*x^q)^(m+1/n+1)*
      ReplaceAll[u,x->(-a*e+c*x^q)^(1/n)/(b*e-d*x^q)^(1/n)]^r,x],x],x,(e*(a+b*x^n)/(c+d*x^n))^(1/q)] /;
    FreeQ[{a,b,c,d,e},x] && PolynomialQ[u,x] && FractionQ[p] && IntegerQ[1/n] && IntegersQ[m,r]
```

$$4. \int u \left(a + b \left(\frac{c}{x} \right)^n \right)^p dx$$

$$1: \int \left(a + b \left(\frac{c}{x} \right)^n \right)^p dx$$

Derivation: Integration by substitution

$$\text{Basis: } F \left[\frac{c}{x} \right] == -c \text{Subst} \left[\frac{F[x]}{x^2}, x, \frac{c}{x} \right] \partial_x \frac{c}{x}$$

Rule 1.5.4.4.1:

$$\int \left(a + b \left(\frac{c}{x} \right)^n \right)^p dx \rightarrow -c \text{Subst} \left[\int \frac{(a + b x^n)^p}{x^2} dx, x, \frac{c}{x} \right]$$

Program code:

```
Int [ (a_. + b_. * (c_./x_)^n_)^p_, x_Symbol] :=
  -c*Subst [Int [ (a+b*x^n)^p/x^2, x], x, c/x] /;
FreeQ [ {a, b, c, n, p}, x]
```

$$2. \int (dx)^m \left(a + b \left(\frac{c}{x} \right)^n \right)^p dx$$

$$1: \int x^m \left(a + b \left(\frac{c}{x} \right)^n \right)^p dx \text{ when } m \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $m \in \mathbb{Z}$, then $x^m F \left[\frac{c}{x} \right] = -c^{m+1} \text{Subst} \left[\frac{F[x]}{x^{m+2}}, x, \frac{c}{x} \right] \partial_x \frac{c}{x}$

Rule 1.5.4.4.2.1: If $m \in \mathbb{Z}$, then

$$\int x^m \left(a + b \left(\frac{c}{x} \right)^n \right)^p dx \rightarrow -c^{m+1} \text{Subst} \left[\int \frac{(a + b x^n)^p}{x^{m+2}} dx, x, \frac{c}{x} \right]$$

Program code:

```
Int [ x^m_.* (a_. + b_. * (c_./x_)^n_)^p_, x_Symbol] :=
  -c^(m+1)*Subst [Int [ (a+b*x^n)^p/x^(m+2), x], x, c/x] /;
FreeQ [ {a, b, c, n, p}, x] && IntegerQ [m]
```

$$2: \int (dx)^m \left(a + b \left(\frac{c}{x} \right)^n \right)^p dx \text{ when } m \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \left((dx)^m \left(\frac{c}{x} \right)^m \right) == 0$$

$$\text{Basis: } F \left[\frac{c}{x} \right] == -c \text{ Subst} \left[\frac{F[x]}{x^2}, x, \frac{c}{x} \right] \partial_x \frac{c}{x}$$

Rule 1.5.4.4.2.2: If $m \notin \mathbb{Z}$, then

$$\int (dx)^m \left(a + b \left(\frac{c}{x} \right)^n \right)^p dx \rightarrow (dx)^m \left(\frac{c}{x} \right)^m \int \frac{\left(a + b \left(\frac{c}{x} \right)^n \right)^p}{\left(\frac{c}{x} \right)^m} dx \rightarrow -c (dx)^m \left(\frac{c}{x} \right)^m \text{Subst} \left[\int \frac{\left(a + b x^n \right)^p}{x^{m+2}} dx, x, \frac{c}{x} \right]$$

Program code:

```
Int[(d.*x_)^m.*(a_.+b_.*(c_/x_)^n_)^p_,x_Symbol] :=
  -c*(d*x)^m*(c/x)^m*Subst[Int[(a+b*x^n)^p/x^(m+2),x],x,c/x] /;
FreeQ[{a,b,c,d,m,n,p},x] && Not[IntegerQ[m]]
```

$$5. \int u \left(a + b \left(\frac{d}{x} \right)^n + c \left(\frac{d}{x} \right)^{2n} \right)^p dx$$

$$1: \int \left(a + b \left(\frac{d}{x} \right)^n + c \left(\frac{d}{x} \right)^{2n} \right)^p dx$$

Derivation: Integration by substitution

$$\text{Basis: } F \left[\frac{d}{x} \right] == -d \text{ Subst} \left[\frac{F[x]}{x^2}, x, \frac{d}{x} \right] \partial_x \frac{d}{x}$$

Rule 1.5.4.5.1:

$$\int \left(a + b \left(\frac{d}{x} \right)^n + c \left(\frac{d}{x} \right)^{2n} \right)^p dx \rightarrow -d \text{ Subst} \left[\int \frac{(a + b x^n + c x^{2n})^p}{x^2} dx, x, \frac{d}{x} \right]$$

Program code:

```
Int[(a_+b_.*(d_/x_)^n+c_.*(d_/x_)^2n_)^p_,x_Symbol] :=
  -d*Subst[Int[(a+b*x^n+c*x^(2*n))^p/x^2,x],x,d/x] /;
FreeQ[{a,b,c,d,n,p},x] && EqQ[n2,2*n]
```

$$2. \int (e x)^m \left(a + b \left(\frac{d}{x} \right)^n + c \left(\frac{d}{x} \right)^{2n} \right)^p dx$$

$$1: \int x^m \left(a + b \left(\frac{d}{x} \right)^n + c \left(\frac{d}{x} \right)^{2n} \right)^p dx \text{ when } m \in \mathbb{Z}$$

Derivation: Integration by substitution

Basis: If $m \in \mathbb{Z}$, then $x^m F \left[\frac{d}{x} \right] == -d^{m+1} \text{Subst} \left[\frac{F[x]}{x^{m+2}}, x, \frac{d}{x} \right] \partial_x \frac{d}{x}$

Rule 1.5.4.5.2.1: If $m \in \mathbb{Z}$, then

$$\int x^m \left(a + b \left(\frac{d}{x} \right)^n + c \left(\frac{d}{x} \right)^{2n} \right)^p dx \rightarrow -d^{m+1} \text{Subst} \left[\int \frac{(a + b x^n + c x^{2n})^p}{x^{m+2}} dx, x, \frac{d}{x} \right]$$

Program code:

```
Int[x^m.*(a+b.*(d./x)^n+c.*(d./x)^2n)^p,x_Symbol] :=
-d^(m+1)*Subst[Int[(a+b*x^n+c*x^(2*n))^p/x^(m+2),x],x,d/x] /;
FreeQ[{a,b,c,d,n,p},x] && EqQ[n2,2*n] && IntegerQ[m]
```

$$2: \int (e x)^m \left(a + b \left(\frac{d}{x} \right)^n + c \left(\frac{d}{x} \right)^{2n} \right)^p dx \text{ when } m \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \left((e x)^m \left(\frac{d}{x} \right)^m \right) == 0$$

$$\text{Basis: } F \left[\frac{d}{x} \right] == -d \text{ Subst} \left[\frac{F[x]}{x^2}, x, \frac{d}{x} \right] \partial_x \frac{d}{x}$$

Rule 1.5.4.5.2.2: If $m \notin \mathbb{Z}$, then

$$\int (e x)^m \left(a + b \left(\frac{d}{x} \right)^n + c \left(\frac{d}{x} \right)^{2n} \right)^p dx \rightarrow (e x)^m \left(\frac{d}{x} \right)^m \int \frac{\left(a + b \left(\frac{d}{x} \right)^n + c \left(\frac{d}{x} \right)^{2n} \right)^p}{\left(\frac{d}{x} \right)^m} dx \rightarrow -d (e x)^m \left(\frac{d}{x} \right)^m \text{Subst} \left[\int \frac{\left(a + b x^n + c x^{2n} \right)^p}{x^{m+2}} dx, x, \frac{d}{x} \right]$$

Program code:

```
Int[(e.*x_)^m.*(a+b.*(d./x_)^n+c.*(d./x_)^n2.)^p_.,x_Symbol] :=
-d*(e*x)^m*(d/x)^m*Subst[Int[(a+b*x^n+c*x^(2*n))^p/x^(m+2),x],x,d/x] /;
FreeQ[{a,b,c,d,e,m,n,p},x] && EqQ[n2,2*n] && Not[IntegerQ[m]]
```

$$6. \int u \left(a + b \left(\frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \text{ when } 2n \in \mathbb{Z}$$

$$1: \int \left(a + b \left(\frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \text{ when } 2n \in \mathbb{Z}$$

Derivation: Integration by substitution

$$\text{Basis: } F \left[\frac{d}{x} \right] == -d \text{ Subst} \left[\frac{F[x]}{x^2}, x, \frac{d}{x} \right] \partial_x \frac{d}{x}$$

Rule 1.5.4.6.1: If $2n \in \mathbb{Z}$, then

$$\int \left(a + b \left(\frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \rightarrow \int \left(a + b \left(\frac{d}{x} \right)^n + \frac{c}{d^{2n}} \left(\frac{d}{x} \right)^{2n} \right)^p dx \rightarrow -d \operatorname{Subst} \left[\int \frac{\left(a + b x^n + \frac{c}{d^{2n}} x^{2n} \right)^p}{x^2} dx, x, \frac{d}{x} \right]$$

Program code:

```
Int [ (a_ + b_.* (d_/x_)^n_ + c_.*x_^n2_)^p_., x_Symbol ] :=
  -d*Subst [ Int [ (a+b*x^n+c/d^(2*n))*x^(2*n) ]^p/x^2, x ], x, d/x ] /;
FreeQ [ {a,b,c,d,n,p}, x ] && EqQ [ n2, -2*n ] && IntegerQ [ 2*n ]
```

2. $\int (e x)^m \left(a + b \left(\frac{d}{x} \right)^n + c x^{-2n} \right)^p dx$ when $2n \in \mathbb{Z}$

1: $\int x^m \left(a + b \left(\frac{d}{x} \right)^n + c x^{-2n} \right)^p dx$ when $2n \in \mathbb{Z} \wedge m \in \mathbb{Z}$

Derivation: Integration by substitution

Basis: If $m \in \mathbb{Z}$, then $x^m F \left[\frac{d}{x} \right] = -d^{m+1} \operatorname{Subst} \left[\frac{F[x]}{x^{m+2}}, x, \frac{d}{x} \right] \partial_x \frac{d}{x}$

Rule 1.5.4.6.2.1: If $2n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, then

$$\int x^m \left(a + b \left(\frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \rightarrow \int x^m \left(a + b \left(\frac{d}{x} \right)^n + \frac{c}{d^{2n}} \left(\frac{d}{x} \right)^{2n} \right)^p dx \rightarrow -d^{m+1} \operatorname{Subst} \left[\int \frac{\left(a + b x^n + \frac{c}{d^{2n}} x^{2n} \right)^p}{x^{m+2}} dx, x, \frac{d}{x} \right]$$

Program code:

```
Int [ x^m_.* (a_ + b_.* (d_/x_)^n_ + c_.*x_^n2_)^p_., x_Symbol ] :=
  -d^(m+1)*Subst [ Int [ (a+b*x^n+c/d^(2*n))*x^(2*n) ]^p/x^(m+2), x ], x, d/x ] /;
FreeQ [ {a,b,c,d,n,p}, x ] && EqQ [ n2, -2*n ] && IntegerQ [ 2*n ] && IntegerQ [ m ]
```

$$2: \int (e x)^m \left(a + b \left(\frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \text{ when } 2n \in \mathbb{Z} \wedge m \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction and integration by substitution

$$\text{Basis: } \partial_x \left((e x)^m \left(\frac{d}{x} \right)^m \right) == 0$$

$$\text{Basis: } F \left[\frac{d}{x} \right] == -d \text{ Subst} \left[\frac{F[x]}{x^2}, x, \frac{d}{x} \right] \partial_x \frac{d}{x}$$

Rule 1.5.4.6.2.2: If $2n \in \mathbb{Z} \wedge m \notin \mathbb{Z}$, then

$$\int (e x)^m \left(a + b \left(\frac{d}{x} \right)^n + c x^{-2n} \right)^p dx \rightarrow (e x)^m \left(\frac{d}{x} \right)^m \int \frac{\left(a + b \left(\frac{d}{x} \right)^n + \frac{c}{d^{2n}} \left(\frac{d}{x} \right)^{2n} \right)^p}{\left(\frac{d}{x} \right)^m} dx \rightarrow -d (e x)^m \left(\frac{d}{x} \right)^m \text{Subst} \left[\int \frac{\left(a + b x^n + \frac{c}{d^{2n}} x^{2n} \right)^p}{x^{m+2}} dx, x, \frac{d}{x} \right]$$

Program code:

```
Int[(e.*x_)^m.*(a+b.*(d./x_)^n+c.*x_^2n.)^p.,x_Symbol] :=
-d*(e*x)^m*(d/x)^m*Subst[Int[(a+b*x^n+c/d^(2*n))*x^(2*n)]^p/x^(m+2),x],x,d/x] /;
FreeQ[{a,b,c,d,e,n,p},x] && EqQ[n2,-2*n] && Not[IntegerQ[m]] && IntegerQ[2*n]
```

7. Binomial products

1. Linear

$$1: \int u^m dx \text{ when } u = a + b x$$

Derivation: Algebraic normalization

Rule: If $u = a + b x$, then

$$\int u^m dx \rightarrow \int (a + b x)^m dx$$

Program code:

```
Int[u_^m_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m,x] /;
  FreeQ[m,x] && LinearQ[u,x] && Not[LinearMatchQ[u,x]]
```

$$2: \int u^m v^n dx \text{ when } u = a + b x \wedge v = c + d x$$

Derivation: Algebraic normalization

Rule: If $u = a + b x \wedge v = c + d x$, then

$$\int u^m v^n dx \rightarrow \int (a + b x)^m (c + d x)^n dx$$

Program code:

```
Int[u_^m_.*v_^n_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^n,x] /;
  FreeQ[{m,n},x] && LinearQ[{u,v},x] && Not[LinearMatchQ[{u,v},x]]
```

$$3: \int u^m v^n w^p dx \text{ when } u = a + bx \wedge v = c + dx \wedge w = e + fx$$

Derivation: Algebraic normalization

Rule: If $u = a + bx \wedge v = c + dx \wedge w = e + fx$, then

$$\int u^m v^n w^p dx \rightarrow \int (a + bx)^m (c + dx)^n (e + fx)^p dx$$

Program code:

```
Int[u_^m_.*v_^n_.*w_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^n*ExpandToSum[w,x]^p,x] /;
  FreeQ[{m,n,p},x] && LinearQ[{u,v,w},x] && Not[LinearMatchQ[{u,v,w},x]]
```

$$4: \int u^m v^n w^p z^q dx \text{ when } u = a + bx \wedge v = c + dx \wedge w = e + fx \wedge z = g + hx$$

Derivation: Algebraic normalization

Rule: If $u = a + bx \wedge v = c + dx \wedge w = e + fx \wedge z = g + hx$, then

$$\int u^m v^n w^p z^q dx \rightarrow \int (a + bx)^m (c + dx)^n (e + fx)^p (g + hx)^q dx$$

Program code:

```
Int[u_^m_.*v_^n_.*w_^p_.*z_^q_,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^n*ExpandToSum[w,x]^p*ExpandToSum[z,x]^q,x] /;
  FreeQ[{m,n,p,q},x] && LinearQ[{u,v,w,z},x] && Not[LinearMatchQ[{u,v,w,z},x]]
```

3. General

1: $\int u^p dx$ when $u = a + b x^n$

Derivation: Algebraic normalization

Rule: If $u = a + b x^n$, then

$$\int u^p dx \rightarrow \int (a + b x^n)^p dx$$

Program code:

```
Int[u_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && BinomialQ[u,x] && Not[BinomialMatchQ[u,x]]
```

2: $\int (c x)^m u^p dx$ when $u = a + b x^n$

Derivation: Algebraic normalization

Rule: If $u = a + b x^n$, then

$$\int (c x)^m u^p dx \rightarrow \int (c x)^m (a + b x^n)^p dx$$

Program code:

```
Int[(c_*x_)^m_*u_^p_,x_Symbol] :=
  Int[(c*x)^m*ExpandToSum[u,x]^p,x] /;
  FreeQ[{c,m,p},x] && BinomialQ[u,x] && Not[BinomialMatchQ[u,x]]
```

$$3: \int u^p v^q dx \text{ when } u = a + b x^n \wedge v = c + d x^n$$

Derivation: Algebraic normalization

Rule: If $u = a + b x^n \wedge v = c + d x^n$, then

$$\int u^p v^q dx \rightarrow \int (a + b x^n)^p (c + d x^n)^q dx$$

Program code:

```
Int[u_^p_.*v_^q_.,x_Symbol] :=
  Int[ExpandToSum[u,x]^p*ExpandToSum[v,x]^q,x] /;
  FreeQ[{p,q},x] && BinomialQ[{u,v},x] && EqQ[BinomialDegree[u,x]-BinomialDegree[v,x],0] && Not[BinomialMatchQ[{u,v},x]]
```

$$4: \int (e x)^m u^p v^q dx \text{ when } u = a + b x^n \wedge v = c + d x^n$$

Derivation: Algebraic normalization

Rule: If $u = a + b x^n \wedge v = c + d x^n$, then

$$\int (e x)^m u^p v^q dx \rightarrow \int (e x)^m (a + b x^n)^p (c + d x^n)^q dx$$

Program code:

```
Int[(e_.*x_)^m_.*u_^p_.*v_^q_.,x_Symbol] :=
  Int[(e*x)^m*ExpandToSum[u,x]^p*ExpandToSum[v,x]^q,x] /;
  FreeQ[{e,m,p,q},x] && BinomialQ[{u,v},x] && EqQ[BinomialDegree[u,x]-BinomialDegree[v,x],0] && Not[BinomialMatchQ[{u,v},x]]
```

$$5: \int u^m v^p w^q dx \text{ when } u = a + b x^n \wedge v = c + d x^n \wedge w = e + f x^n$$

Derivation: Algebraic normalization

Rule: If $u = a + b x^n \wedge v = c + d x^n \wedge w = e + f x^n$, then

$$\int u^m v^p w^q dx \rightarrow \int (a + b x^n)^m (c + d x^n)^p (e + f x^n)^q dx$$

Program code:

```
Int[u_^m_.*v_^p_.*w_^q_.,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^p*ExpandToSum[w,x]^q,x] /;
FreeQ[{m,p,q},x] && BinomialQ[{u,v,w},x] && EqQ[BinomialDegree[u,x]-BinomialDegree[v,x],0] &&
EqQ[BinomialDegree[u,x]-BinomialDegree[w,x],0] && Not[BinomialMatchQ[{u,v,w},x]]
```

$$6: \int (g x)^m u^p v^q z^r dx \text{ when } u = a + b x^n \wedge v = c + d x^n \wedge z = e + f x^n$$

Derivation: Algebraic normalization

Rule: If $u = a + b x^n \wedge v = c + d x^n \wedge z = e + f x^n$, then

$$\int (g x)^m u^p v^q z^r dx \rightarrow \int (g x)^m (a + b x^n)^p (c + d x^n)^q (e + f x^n)^r dx$$

Program code:

```
Int[(g_.*x_)^m_.*u_^p_.*v_^q_.*z_^r_.,x_Symbol] :=
  Int[(g*x)^m*ExpandToSum[u,x]^p*ExpandToSum[v,x]^q*ExpandToSum[z,x]^r,x] /;
FreeQ[{g,m,p,q,r},x] && BinomialQ[{u,v,z},x] && EqQ[BinomialDegree[u,x]-BinomialDegree[v,x],0] &&
EqQ[BinomialDegree[u,x]-BinomialDegree[z,x],0] && Not[BinomialMatchQ[{u,v,z},x]]
```

$$7: \int (c x)^m P_q[x] u^p dx \text{ when } u = a + b x^n$$

Derivation: Algebraic normalization

Rule: If $u = a + b x^n$, then

$$\int (c x)^m P_q[x] u^p dx \rightarrow \int (c x)^m P_q[x] (a + b x^n)^p dx$$

Program code:

```
Int[(c_*x_)^m_*Pq_*u_^p_,x_Symbol] :=
  Int[(c*x)^m*Pq*ExpandToSum[u,x]^p,x] /;
  FreeQ[{c,m,p},x] && PolyQ[Pq,x] && BinomialQ[u,x] && Not[BinomialMatchQ[u,x]]
```

4. Improper

$$1: \int u^p dx \text{ when } u = a x^j + b x^n$$

Derivation: Algebraic normalization

Rule: If $u = a x^j + b x^n$, then

$$\int u^p dx \rightarrow \int (a x^j + b x^n)^p dx$$

Program code:

```
Int[u_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && GeneralizedBinomialQ[u,x] && Not[GeneralizedBinomialMatchQ[u,x]]
```

$$2: \int (c x)^m u^p dx \text{ when } u = a x^j + b x^n$$

Derivation: Algebraic normalization

Rule: If $u = a x^j + b x^n$, then

$$\int (c x)^m u^p dx \rightarrow \int (c x)^m (a x^j + b x^n)^p dx$$

Program code:

```
Int[(c.*x_)^m_.*u^p_,x_Symbol] :=
  Int[(c*x)^m*ExpandToSum[u,x]^p,x] /;
  FreeQ[{c,m,p},x] && GeneralizedBinomialQ[u,x] && Not[GeneralizedBinomialMatchQ[u,x]]
```

8 Trinomial products

1. Quadratic

$$1: \int u^p dx \text{ when } u = a + b x + c x^2$$

Derivation: Algebraic normalization

Rule: If $u = a + b x + c x^2$, then

$$\int u^p dx \rightarrow \int (a + b x + c x^2)^p dx$$

Program code:

```
Int[u^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && QuadraticQ[u,x] && Not[QuadraticMatchQ[u,x]]
```

$$2: \int u^m v^p dx \text{ when } u = d + ex \wedge v = a + bx + cx^2$$

Derivation: Algebraic normalization

Rule: If $u = d + ex \wedge v = a + bx + cx^2$, then

$$\int u^m v^p dx \rightarrow \int (d + ex)^m (a + bx + cx^2)^p dx$$

Program code:

```
Int[u_^m_.*v_^p_.,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^p,x] /;
  FreeQ[{m,p},x] && LinearQ[u,x] && QuadraticQ[v,x] && Not[LinearMatchQ[u,x] && QuadraticMatchQ[v,x]]
```

$$3: \int u^m v^n w^p dx \text{ when } u = d + ex \wedge v = f + gx \wedge w = a + bx + cx^2$$

Derivation: Algebraic normalization

Rule: If $u = d + ex \wedge v = f + gx \wedge w = a + bx + cx^2$, then

$$\int u^m v^n w^p dx \rightarrow \int (d + ex)^m (f + gx)^n (a + bx + cx^2)^p dx$$

Program code:

```
Int[u_^m_.*v_^n_.*w_^p_.,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*ExpandToSum[v,x]^n*ExpandToSum[w,x]^p,x] /;
  FreeQ[{m,n,p},x] && LinearQ[{u,v},x] && QuadraticQ[w,x] && Not[LinearMatchQ[{u,v},x] && QuadraticMatchQ[w,x]]
```

$$4: \int u^p v^q dx \text{ when } u = a + b x + c x^2 \wedge v = d + e x + f x^2$$

Derivation: Algebraic normalization

Rule: If $u = a + b x + c x^2 \wedge v = d + e x + f x^2$, then

$$\int u^p v^q dx \rightarrow \int (a + b x + c x^2)^p (d + e x + f x^2)^q dx$$

Program code:

```
Int[u_^p_.*v_^q_.,x_Symbol] :=
  Int[ExpandToSum[u,x]^p*ExpandToSum[v,x]^q,x] /;
  FreeQ[{p,q},x] && QuadraticQ[{u,v},x] && Not[QuadraticMatchQ[{u,v},x]]
```

$$5: \int z^m u^p v^q dx \text{ when } z = g + h x \wedge u = a + b x + c x^2 \wedge v = d + e x + f x^2$$

Derivation: Algebraic normalization

Note: This normalization needs to be done before trying polynomial integration rules.

Rule 1.2.1.5.N: If $z = g + h x \wedge u = a + b x + c x^2 \wedge v = d + e x + f x^2$, then

$$\int z^m u^p v^q dx \rightarrow \int (g + h x)^m (a + b x + c x^2)^p (d + e x + f x^2)^q dx$$

Program code:

```
Int[z_^m_.*u_^p_.*v_^q_.,x_Symbol] :=
  Int[ExpandToSum[z,x]^m*ExpandToSum[u,x]^p*ExpandToSum[v,x]^q,x] /;
  FreeQ[{m,p,q},x] && LinearQ[z,x] && QuadraticQ[{u,v},x] && Not[LinearMatchQ[z,x] && QuadraticMatchQ[{u,v},x]]
```

6: $\int P_q[x] u^p dx$ when $u = a + b x + c x^2$

Derivation: Algebraic normalization

Rule: If $u = a + b x + c x^2$, then

$$\int P_q[x] u^p dx \rightarrow \int P_q[x] (a + b x + c x^2)^p dx$$

Program code:

```
Int[Pq_*u^p_.,x_Symbol] :=
  Int[Pq*ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && PolyQ[Pq,x] && QuadraticQ[u,x] && Not[QuadraticMatchQ[u,x]]
```

7: $\int u^m P_q[x] v^p dx$ when $u = d + e x$ \wedge $v = a + b x + c x^2$

Derivation: Algebraic normalization

Rule: If $u = d + e x$ \wedge $v = a + b x + c x^2$, then

$$\int u^m P_q[x] v^p dx \rightarrow \int (d + e x)^m P_q[x] (a + b x + c x^2)^p dx$$

Program code:

```
Int[u^m_.*Pq_*v^p_.,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*Pq*ExpandToSum[v,x]^p,x] /;
  FreeQ[{m,p},x] && PolyQ[Pq,x] && LinearQ[u,x] && QuadraticQ[v,x] && Not[LinearMatchQ[u,x] && QuadraticMatchQ[v,x]]
```

3. General

1: $\int u^p dx$ when $u = a + b x^n + c x^{2n}$

Derivation: Algebraic normalization

Rule: If $u = a + b x^n + c x^{2n}$, then

$$\int u^p dx \rightarrow \int (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[u_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && TrinomialQ[u,x] && Not[TrinomialMatchQ[u,x]]
```

2: $\int (dx)^m u^p dx$ when $u = a + b x^n + c x^{2n}$

Derivation: Algebraic normalization

Rule: If $u = a + b x^n + c x^{2n}$, then

$$\int (dx)^m u^p dx \rightarrow \int (dx)^m (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[(d_*x_)^m_*u_^p_,x_Symbol] :=
  Int[(d*x)^m*ExpandToSum[u,x]^p,x] /;
  FreeQ[{d,m,p},x] && TrinomialQ[u,x] && Not[TrinomialMatchQ[u,x]]
```

$$3: \int u^q v^p dx \text{ when } u = d + e x^n \wedge v = a + b x^n + c x^{2n}$$

Derivation: Algebraic normalization

Rule: If $u = d + e x^n \wedge v = a + b x^n + c x^{2n}$, then

$$\int u^q v^p dx \rightarrow \int (d + e x^n)^q (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[u_^q_.*v_^p_.,x_Symbol] :=
  Int[ExpandToSum[u,x]^q*ExpandToSum[v,x]^p,x] /;
  FreeQ[{p,q},x] && BinomialQ[u,x] && TrinomialQ[v,x] && Not[BinomialMatchQ[u,x] && TrinomialMatchQ[v,x]]
```

```
Int[u_^q_.*v_^p_.,x_Symbol] :=
  Int[ExpandToSum[u,x]^q*ExpandToSum[v,x]^p,x] /;
  FreeQ[{p,q},x] && BinomialQ[u,x] && BinomialQ[v,x] && Not[BinomialMatchQ[u,x] && BinomialMatchQ[v,x]]
```

$$4: \int (f x)^m z^q u^p dx \text{ when } z = d + e x^n \wedge u = a + b x^n + c x^{2n}$$

Derivation: Algebraic normalization

Rule: If $z = d + e x^n \wedge u = a + b x^n + c x^{2n}$, then

$$\int (f x)^m z^q u^p dx \rightarrow \int (f x)^m (d + e x^n)^q (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int[(f_.*x_)^m_.*z_^q_.*u_^p_.,x_Symbol] :=
  Int[(f*x)^m*ExpandToSum[z,x]^q*ExpandToSum[u,x]^p,x] /;
  FreeQ[{f,m,p,q},x] && BinomialQ[z,x] && TrinomialQ[u,x] && Not[BinomialMatchQ[z,x] && TrinomialMatchQ[u,x]]
```

```
Int [(f_.*x_)^m_.*z_^q_.*u_^p_.,x_Symbol] :=
  Int [(f*x)^m*ExpandToSum[z,x]^q*ExpandToSum[u,x]^p,x] /;
FreeQ[{f,m,p,q},x] && BinomialQ[z,x] && BinomialQ[u,x] && Not[BinomialMatchQ[z,x] && BinomialMatchQ[u,x]]
```

5: $\int P_q[x] u^p dx$ when $u = a + b x^n + c x^{2n}$

Derivation: Algebraic normalization

Rule: If $u = a + b x^n + c x^{2n}$, then

$$\int P_q[x] u^p dx \rightarrow \int P_q[x] (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int [Pq_*u_^p_.,x_Symbol] :=
  Int [Pq*ExpandToSum[u,x]^p,x] /;
FreeQ[p,x] && PolyQ[Pq,x] && TrinomialQ[u,x] && Not[TrinomialMatchQ[u,x]]
```

6: $\int (dx)^m P_q[x] u^p dx$ when $u = a + b x^n + c x^{2n}$

Derivation: Algebraic normalization

Rule: If $u = a + b x^n + c x^{2n}$, then

$$\int (dx)^m P_q[x] u^p dx \rightarrow \int (dx)^m P_q[x] (a + b x^n + c x^{2n})^p dx$$

Program code:

```
Int [(d_.*x_)^m_.*Pq_*u_^p_.,x_Symbol] :=
  Int [(d*x)^m*Pq*ExpandToSum[u,x]^p,x] /;
FreeQ[{d,m,p},x] && PolyQ[Pq,x] && TrinomialQ[u,x] && Not[TrinomialMatchQ[u,x]]
```

4. Improper

1: $\int u^p dx$ when $u = a x^q + b x^n + c x^{2n-q}$

Derivation: Algebraic normalization

Rule: If $u = a x^q + b x^n + c x^{2n-q}$, then

$$\int u^p dx \rightarrow \int (a x^q + b x^n + c x^{2n-q})^p dx$$

Program code:

```
Int[u_^p_,x_Symbol] :=
  Int[ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && GeneralizedTrinomialQ[u,x] && Not[GeneralizedTrinomialMatchQ[u,x]]
```

2: $\int (dx)^m u^p dx$ when $u = a x^q + b x^n + c x^{2n-q}$

Derivation: Algebraic normalization

Rule: If $u = a x^q + b x^n + c x^{2n-q}$, then

$$\int (dx)^m u^p dx \rightarrow \int (dx)^m (a x^q + b x^n + c x^{2n-q})^p dx$$

Program code:

```
Int[(d.*x_)^m_.*u_^p_,x_Symbol] :=
  Int[(d*x)^m*ExpandToSum[u,x]^p,x] /;
  FreeQ[{d,m,p},x] && GeneralizedTrinomialQ[u,x] && Not[GeneralizedTrinomialMatchQ[u,x]]
```

$$3: \int z u^p dx \text{ when } z = A + B x^{n-q} \wedge u = a x^q + b x^n + c x^{2n-q}$$

Derivation: Algebraic normalization

Rule: If $z = A + B x^{n-q} \wedge u = a x^q + b x^n + c x^{2n-q}$, then

$$\int z u^p dx \rightarrow \int (A + B x^{n-q}) (a x^q + b x^n + c x^{2n-q})^p dx$$

Program code:

```
Int[z_*u_^p_.,x_Symbol] :=
  Int[ExpandToSum[z,x]*ExpandToSum[u,x]^p,x] /;
  FreeQ[p,x] && BinomialQ[z,x] && GeneralizedTrinomialQ[u,x] &&
  EqQ[BinomialDegree[z,x]-GeneralizedTrinomialDegree[u,x],0] && Not[BinomialMatchQ[z,x] && GeneralizedTrinomialMatchQ[u,x]]
```

$$4: \int (f x)^m z u^p dx \text{ when } z = A + B x^{n-q} \wedge u = a x^q + b x^n + c x^{2n-q}$$

Derivation: Algebraic normalization

Rule: If $z = A + B x^{n-q} \wedge u = a x^q + b x^n + c x^{2n-q}$, then

$$\int (f x)^m z u^p dx \rightarrow \int (f x)^m (A + B x^{n-q}) (a x^q + b x^n + c x^{2n-q})^p dx$$

Program code:

```
Int[(f_*x_)^m_*z_*u_^p_.,x_Symbol] :=
  Int[(f*x)^m*ExpandToSum[z,x]*ExpandToSum[u,x]^p,x] /;
  FreeQ[{f,m,p},x] && BinomialQ[z,x] && GeneralizedTrinomialQ[u,x] &&
  EqQ[BinomialDegree[z,x]-GeneralizedTrinomialDegree[u,x],0] && Not[BinomialMatchQ[z,x] && GeneralizedTrinomialMatchQ[u,x]]
```

