

# Charlwood Integration Test Suite

The following is a list of the 50 example integration problems from Kevin Charlwood's 2008 article *Integration on Computer Algebra Systems*. Each integral along with its optimal antiderivative (that is, the best antiderivative found so far) is shown. These problems are also available in machine readable form from the *Rubi* home page expressed in *Axiom*, *Maple*, *Mathematica* and *Maxima* syntax.

## ■ Problem #1

$$\int \text{ArcSin}[x] \text{Log}[x] \, dx = -2 \sqrt{1-x^2} + \text{ArcTanh}[\sqrt{1-x^2}] - x \text{ArcSin}[x] (1 - \text{Log}[x]) + \sqrt{1-x^2} \text{Log}[x]$$

## ■ Problem #2

$$\int \frac{x \text{ArcSin}[x]}{\sqrt{1-x^2}} \, dx = x - \sqrt{1-x^2} \text{ArcSin}[x]$$

## ■ Problem #3

$$\int \text{ArcSin}[\sqrt{x+1} - \sqrt{x}] \, dx = \frac{(\sqrt{x} + 3\sqrt{1+x}) \sqrt{-x + \sqrt{x} \sqrt{1+x}}}{4\sqrt{2}} - \left(\frac{3}{8} + x\right) \text{ArcSin}[\sqrt{x} - \sqrt{1+x}]$$

## ■ Problem #4

$$\int \text{Log}[1 + x \sqrt{1+x^2}] \, dx = -2x + \sqrt{2(1+\sqrt{5})} \text{ArcTan}\left[\sqrt{-2+\sqrt{5}} \left(x + \sqrt{1+x^2}\right)\right] - \sqrt{2(-1+\sqrt{5})} \text{ArcTanh}\left[\sqrt{2+\sqrt{5}} \left(x + \sqrt{1+x^2}\right)\right] + x \text{Log}[1 + x \sqrt{1+x^2}]$$

## ■ Problem #5

$$\int \frac{\text{Cos}[x]^2}{\sqrt{\text{Cos}[x]^4 + \text{Cos}[x]^2 + 1}} \, dx = \frac{x}{3} + \frac{1}{3} \text{ArcTan}\left[\frac{\text{Cos}[x] (1 + \text{Cos}[x]^2) \text{Sin}[x]}{1 + \text{Cos}[x]^2 \sqrt{1 + \text{Cos}[x]^2 + \text{Cos}[x]^4}}\right]$$

■ **Problem #6**

$$\int \tan[x] \sqrt{1 + \tan[x]^4} \, dx = -\frac{1}{2} \operatorname{ArcSinh}[\tan[x]^2] - \frac{\operatorname{ArcTanh}\left[\frac{1 - \tan[x]^2}{\sqrt{2} \sqrt{1 + \tan[x]^4}}\right]}{\sqrt{2}} + \frac{1}{2} \sqrt{1 + \tan[x]^4}$$

■ **Problem #7**

$$\int \frac{\tan[x]}{\sqrt{1 + \sec[x]^3}} \, dx = -\frac{2}{3} \operatorname{ArcTanh}\left[\sqrt{1 + \sec[x]^3}\right]$$

■ **Problem #8**

$$\int \sqrt{\tan[x]^2 + 2 \tan[x] + 2} \, dx = \operatorname{ArcSinh}[1 + \tan[x]] - \sqrt{\frac{1}{2} (1 + \sqrt{5})} \operatorname{ArcTan}\left[\frac{2 - (1 + \sqrt{5}) \tan[x]}{\sqrt{2} (1 + \sqrt{5}) \sqrt{2 + 2 \tan[x] + \tan[x]^2}}\right] - \sqrt{\frac{1}{2} (-1 + \sqrt{5})} \operatorname{ArcTanh}\left[\frac{2 - (1 - \sqrt{5}) \tan[x]}{\sqrt{2} (-1 + \sqrt{5}) \sqrt{2 + 2 \tan[x] + \tan[x]^2}}\right]$$

■ **Problem #9**

$$\int \sin[x] \operatorname{ArcTan}\left[\sqrt{\sec[x] - 1}\right] \, dx = \frac{1}{2} \operatorname{ArcTan}\left[\sqrt{-1 + \sec[x]}\right] - \operatorname{ArcTan}\left[\sqrt{-1 + \sec[x]}\right] \cos[x] + \frac{1}{2} \cos[x] \sqrt{-1 + \sec[x]}$$

■ **Problem #10**

$$\int \frac{x^3 e^{\operatorname{ArcSin}[x]}}{\sqrt{1 - x^2}} \, dx = \frac{1}{10} e^{\operatorname{ArcSin}[x]} \left( 3x + x^3 - 3\sqrt{1 - x^2} - 3x^2 \sqrt{1 - x^2} \right)$$

■ **Problem #11**

$$\int \frac{x \operatorname{Log}[1 + x^2] \operatorname{Log}\left[x + \sqrt{1 + x^2}\right]}{\sqrt{1 + x^2}} \, dx = 4x - 2 \operatorname{ArcTan}[x] - 2 \sqrt{1 + x^2} \operatorname{Log}\left[x + \sqrt{1 + x^2}\right] + \operatorname{Log}[1 + x^2] \left( -x + \sqrt{1 + x^2} \operatorname{Log}\left[x + \sqrt{1 + x^2}\right] \right)$$

■ **Problem #12**

$$\int \text{ArcTan}\left[x + \sqrt{1-x^2}\right] dx =$$

$$-\frac{\text{ArcSin}[x]}{2} + \frac{1}{4}\sqrt{3} \text{ArcTan}\left[\frac{-1+\sqrt{3}x}{\sqrt{1-x^2}}\right] + \frac{1}{4}\sqrt{3} \text{ArcTan}\left[\frac{1+\sqrt{3}x}{\sqrt{1-x^2}}\right] - \frac{1}{4}\sqrt{3} \text{ArcTan}\left[\frac{-1+2x^2}{\sqrt{3}}\right] +$$

$$x \text{ArcTan}\left[x + \sqrt{1-x^2}\right] - \frac{1}{4} \text{ArcTanh}\left[x\sqrt{1-x^2}\right] - \frac{1}{8} \text{Log}\left[1-x^2+x^4\right]$$

■ **Problem #13**

$$\int \frac{x \text{ArcTan}\left[x + \sqrt{1-x^2}\right]}{\sqrt{1-x^2}} dx =$$

$$-\frac{\text{ArcSin}[x]}{2} + \frac{1}{4}\sqrt{3} \text{ArcTan}\left[\frac{-1+\sqrt{3}x}{\sqrt{1-x^2}}\right] + \frac{1}{4}\sqrt{3} \text{ArcTan}\left[\frac{1+\sqrt{3}x}{\sqrt{1-x^2}}\right] - \frac{1}{4}\sqrt{3} \text{ArcTan}\left[\frac{-1+2x^2}{\sqrt{3}}\right] -$$

$$\sqrt{1-x^2} \text{ArcTan}\left[x + \sqrt{1-x^2}\right] + \frac{1}{4} \text{ArcTanh}\left[x\sqrt{1-x^2}\right] + \frac{1}{8} \text{Log}\left[1-x^2+x^4\right]$$

■ **Problem #14**

$$\int \frac{\text{ArcSin}[x]}{1+\sqrt{1-x^2}} dx = -\frac{x \text{ArcSin}[x]}{1+\sqrt{1-x^2}} + \frac{\text{ArcSin}[x]^2}{2} - \text{Log}\left[1+\sqrt{1-x^2}\right]$$

■ **Problem #15**

$$\int \frac{\text{Log}\left[x + \sqrt{1+x^2}\right]}{(1-x^2)^{3/2}} dx = -\frac{1}{2} \text{ArcSin}[x^2] + \frac{x \text{Log}\left[x + \sqrt{1+x^2}\right]}{\sqrt{1-x^2}}$$

■ **Problem #16**

$$\int \frac{\text{ArcSin}[x]}{(1+x^2)^{3/2}} dx = \frac{x \text{ArcSin}[x]}{\sqrt{1+x^2}} - \frac{\text{ArcSin}[x^2]}{2}$$

## ■ Problem #17

$$\int \frac{\text{Log}\left[x + \sqrt{x^2 - 1}\right]}{(1 + x^2)^{3/2}} dx = -\frac{1}{2} \text{ArcCosh}\left[x^2\right] + \frac{x \text{Log}\left[x + \sqrt{-1 + x^2}\right]}{\sqrt{1 + x^2}}$$

## ■ Problem #18

$$\int \frac{\text{Log}[x]}{x^2 \sqrt{x^2 - 1}} dx = \frac{\sqrt{-1 + x^2}}{x} - \text{ArcTanh}\left[\frac{x}{\sqrt{-1 + x^2}}\right] + \frac{\sqrt{-1 + x^2} \text{Log}[x]}{x}$$

## ■ Problem #19

$$\int \frac{\sqrt{1 + x^3}}{x} dx = \frac{2\sqrt{1 + x^3}}{3} - \frac{2}{3} \text{ArcTanh}\left[\sqrt{1 + x^3}\right]$$

## ■ Problem #20

$$\int \frac{x \text{Log}\left[x + \sqrt{x^2 - 1}\right]}{\sqrt{x^2 - 1}} dx = -x + \sqrt{-1 + x^2} \text{Log}\left[x + \sqrt{-1 + x^2}\right]$$

## ■ Problem #21

$$\int \frac{x^3 \text{ArcSin}[x]}{\sqrt{1 - x^4}} dx = \frac{1}{4} x \sqrt{1 + x^2} - \frac{1}{2} \sqrt{1 - x^4} \text{ArcSin}[x] + \frac{\text{ArcSinh}[x]}{4}$$

## ■ Problem #22

$$\int \frac{x^3 \text{ArcSec}[x]}{\sqrt{x^4 - 1}} dx = -\frac{\sqrt{-1 + x^4}}{2\sqrt{1 - \frac{1}{x^2}} x} + \frac{1}{2} \sqrt{-1 + x^4} \text{ArcSec}[x] + \frac{1}{2} \text{ArcTanh}\left[\frac{\sqrt{1 - \frac{1}{x^2}} x}{\sqrt{-1 + x^4}}\right]$$

## ■ Problem #23

$$\int \frac{x \text{ArcTan}[x] \text{Log}\left[x + \sqrt{1 + x^2}\right]}{\sqrt{1 + x^2}} dx = -x \text{ArcTan}[x] + \frac{1}{2} \text{Log}\left[1 + x^2\right] + \sqrt{1 + x^2} \text{ArcTan}[x] \text{Log}\left[x + \sqrt{1 + x^2}\right] - \frac{1}{2} \text{Log}\left[x + \sqrt{1 + x^2}\right]^2$$

■ **Problem #24**

$$\int \frac{x \operatorname{Log}[1 + \sqrt{1 - x^2}]}{\sqrt{1 - x^2}} dx = \sqrt{1 - x^2} - \operatorname{Log}[1 + \sqrt{1 - x^2}] - \sqrt{1 - x^2} \operatorname{Log}[1 + \sqrt{1 - x^2}]$$

■ **Problem #25**

$$\int \frac{x \operatorname{Log}[x + \sqrt{1 + x^2}]}{\sqrt{1 + x^2}} dx = -x + \sqrt{1 + x^2} \operatorname{Log}[x + \sqrt{1 + x^2}]$$

■ **Problem #26**

$$\int \frac{x \operatorname{Log}[x + \sqrt{1 - x^2}]}{\sqrt{1 - x^2}} dx = \sqrt{1 - x^2} + \frac{\operatorname{ArcTanh}[\sqrt{2} x]}{\sqrt{2}} - \frac{\operatorname{ArcTanh}[\sqrt{2} \sqrt{1 - x^2}]}{\sqrt{2}} - \sqrt{1 - x^2} \operatorname{Log}[x + \sqrt{1 - x^2}]$$

■ **Problem #27**

$$\int \frac{\operatorname{Log}[x]}{x^2 \sqrt{1 - x^2}} dx = -\frac{\sqrt{1 - x^2}}{x} - \operatorname{ArcSin}[x] - \frac{\sqrt{1 - x^2} \operatorname{Log}[x]}{x}$$

■ **Problem #28**

$$\int \frac{x \operatorname{ArcTan}[x]}{\sqrt{1 + x^2}} dx = -\operatorname{ArcSinh}[x] + \sqrt{1 + x^2} \operatorname{ArcTan}[x]$$

■ **Problem #29**

$$\int \frac{\operatorname{ArcTan}[x]}{x^2 \sqrt{1 - x^2}} dx = -\frac{\sqrt{1 - x^2} \operatorname{ArcTan}[x]}{x} - \operatorname{ArcTanh}[\sqrt{1 - x^2}] + \sqrt{2} \operatorname{ArcTanh}\left[\frac{\sqrt{1 - x^2}}{\sqrt{2}}\right]$$

■ **Problem #30**

$$\int \frac{x \operatorname{ArcTan}[x]}{\sqrt{1 - x^2}} dx = -\operatorname{ArcSin}[x] - \sqrt{1 - x^2} \operatorname{ArcTan}[x] + \sqrt{2} \operatorname{ArcTan}\left[\frac{\sqrt{2} x}{\sqrt{1 - x^2}}\right]$$

■ **Problem #31**

$$\int \frac{\text{ArcTan}[x]}{x^2 \sqrt{1+x^2}} dx = -\frac{\sqrt{1+x^2} \text{ArcTan}[x]}{x} - \text{ArcTanh}\left[\sqrt{1+x^2}\right]$$

■ **Problem #32**

$$\int \frac{\text{ArcSin}[x]}{x^2 \sqrt{1-x^2}} dx = -\frac{\sqrt{1-x^2} \text{ArcSin}[x]}{x} + \text{Log}[x]$$

■ **Problem #33**

$$\int \frac{x \text{Log}[x]}{\sqrt{x^2-1}} dx = -\sqrt{x^2-1} + \text{ArcTan}\left[\sqrt{x^2-1}\right] + \sqrt{x^2-1} \text{Log}[x]$$

■ **Problem #34**

$$\int \frac{\text{Log}[x]}{x^2 \sqrt{1+x^2}} dx = -\frac{\sqrt{1+x^2}}{x} + \text{ArcSinh}[x] - \frac{\sqrt{1+x^2} \text{Log}[x]}{x}$$

■ **Problem #35**

$$\int \frac{x \text{ArcSec}[x]}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} \text{ArcSec}[x] - \frac{x \text{Log}[x]}{\sqrt{x^2-1}}$$

■ **Problem #36**

$$\int \frac{x \text{Log}[x]}{\sqrt{1+x^2}} dx = -\sqrt{1+x^2} + \text{ArcTanh}\left[\sqrt{1+x^2}\right] + \sqrt{1+x^2} \text{Log}[x]$$

■ **Problem #37**

$$\int \frac{\sin[x]}{1+\sin[x]^2} dx = -\frac{\text{ArcTanh}\left[\frac{\cos[x]}{\sqrt{2}}\right]}{\sqrt{2}}$$

■ **Problem #38**

$$\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}} dx = \frac{\text{ArcTanh}\left[\frac{\sqrt{2}x}{\sqrt{1+x^4}}\right]}{\sqrt{2}}$$

■ **Problem #39**

$$\int \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}} dx = \frac{\text{ArcTan}\left[\frac{\sqrt{2}x}{\sqrt{1+x^4}}\right]}{\sqrt{2}}$$

■ **Problem #40**

$$\int \frac{\text{Log}[\text{Sin}[x]]}{1+\text{Sin}[x]} dx = -x - \text{ArcTanh}[\text{Cos}[x]] - \frac{\text{Cos}[x] \text{Log}[\text{Sin}[x]]}{1+\text{Sin}[x]}$$

■ **Problem #41**

$$\int \text{Log}[\text{Sin}[x]] \sqrt{1+\text{Sin}[x]} dx = \frac{4 \text{Cos}[x]}{\sqrt{1+\text{Sin}[x]}} - \frac{2 \text{Cos}[x] \text{Log}[\text{Sin}[x]]}{\sqrt{1+\text{Sin}[x]}} - 4 \text{ArcTanh}\left[\frac{\text{Cos}[x]}{\sqrt{1+\text{Sin}[x]}}\right]$$

■ **Problem #42**

$$\int \frac{\text{Sec}[x]}{\sqrt{\text{Sec}[x]^4 - 1}} dx = -\frac{\text{ArcTanh}\left[\frac{\sqrt{\text{Sec}[x]^4 - 1}}{\sqrt{2} \text{Sec}[x] \text{Tan}[x]}\right]}{\sqrt{2}}$$

■ **Problem #43**

$$\int \frac{\text{Tan}[x]}{\sqrt{1+\text{Tan}[x]^4}} dx = -\frac{\text{ArcTanh}\left[\frac{1-\text{Tan}[x]^2}{\sqrt{2} \sqrt{1+\text{Tan}[x]^4}}\right]}{2\sqrt{2}}$$

**■ Problem #44**

$$\int \frac{\sin[x]}{\sqrt{1 - \sin[x]^6}} dx = \frac{\text{ArcTanh}\left[\frac{\sqrt{3} \cos[x] (1 + \sin[x]^2)}{2\sqrt{1 - \sin[x]^6}}\right]}{2\sqrt{3}}$$

**■ Problem #45**

$$\int \sqrt{\sqrt{\sec[x] + 1} - \sqrt{\sec[x] - 1}} dx =$$

$$\sqrt{2} \cot[x] \sqrt{-1 + \sec[x]} \sqrt{1 + \sec[x]} \left( \sqrt{-1 + \sqrt{2}} \text{ArcTan}\left[\frac{\sqrt{-2 + 2\sqrt{2}} (-\sqrt{2} - \sqrt{-1 + \sec[x]} + \sqrt{1 + \sec[x]})}{2\sqrt{-\sqrt{-1 + \sec[x]} + \sqrt{1 + \sec[x]}}}\right] \right)$$

$$- \sqrt{1 + \sqrt{2}} \text{ArcTan}\left[\frac{\sqrt{2 + 2\sqrt{2}} (-\sqrt{2} - \sqrt{-1 + \sec[x]} + \sqrt{1 + \sec[x]})}{2\sqrt{-\sqrt{-1 + \sec[x]} + \sqrt{1 + \sec[x]}}}\right] - \sqrt{1 + \sqrt{2}} \text{ArcTanh}\left[\frac{\sqrt{-2 + 2\sqrt{2}} \sqrt{-\sqrt{-1 + \sec[x]} + \sqrt{1 + \sec[x]}}}{\sqrt{2} - \sqrt{-1 + \sec[x]} + \sqrt{1 + \sec[x]}}\right] +$$

$$\left. \sqrt{-1 + \sqrt{2}} \text{ArcTanh}\left[\frac{\sqrt{2 + 2\sqrt{2}} \sqrt{-\sqrt{-1 + \sec[x]} + \sqrt{1 + \sec[x]}}}{\sqrt{2} - \sqrt{-1 + \sec[x]} + \sqrt{1 + \sec[x]}}\right] \right)$$

**■ Problem #46**

$$\int x \text{Log}[x^2 + 1] \text{ArcTan}[x]^2 dx = x \text{ArcTan}[x] (3 - \text{Log}[1 + x^2]) - \frac{1}{4} (6 - \text{Log}[1 + x^2]) \text{Log}[1 + x^2] - \frac{1}{2} \text{ArcTan}[x]^2 (3 + x^2 - (1 + x^2) \text{Log}[1 + x^2])$$

**■ Problem #47**

$$\int \text{ArcTan}\left[x \sqrt{1 + x^2}\right] dx =$$

$$x \text{ArcTan}\left[x \sqrt{1 + x^2}\right] + \frac{1}{2} \text{ArcTan}\left[\sqrt{3} - 2\sqrt{1 + x^2}\right] - \frac{1}{2} \text{ArcTan}\left[\sqrt{3} + 2\sqrt{1 + x^2}\right] - \frac{1}{4} \sqrt{3} \text{Log}\left[2 + x^2 - \sqrt{3} \sqrt{1 + x^2}\right] + \frac{1}{4} \sqrt{3} \text{Log}\left[2 + x^2 + \sqrt{3} \sqrt{1 + x^2}\right]$$

**■ Problem #48**

$$\int \text{ArcTan}\left[\sqrt{x+1} - \sqrt{x}\right] dx = \frac{\sqrt{x}}{2} + (1+x) \text{ArcTan}\left[\sqrt{1+x} - \sqrt{x}\right]$$



- **Problem #49 (Note: The problem was altered so as to have an answer involving only elementary functions and operators.)**

$$\int \text{ArcSin}\left[\frac{x}{\sqrt{1-x^2}}\right] dx = x \text{ArcSin}\left[\frac{x}{\sqrt{1-x^2}}\right] + \text{ArcTan}\left[\sqrt{1-2x^2}\right]$$

- **Problem #50**

$$\int \text{ArcTan}\left[x \sqrt{1-x^2}\right] dx =$$

$$x \text{ArcTan}\left[x \sqrt{1-x^2}\right] - \sqrt{\frac{1}{2}(1+\sqrt{5})} \text{ArcTan}\left[\sqrt{\frac{1}{2}(1+\sqrt{5})} \sqrt{1-x^2}\right] + \sqrt{\frac{1}{2}(-1+\sqrt{5})} \text{ArcTan}\left[\sqrt{\frac{1}{2}(-1+\sqrt{5})} \sqrt{1-x^2}\right]$$